MATH 54 Lecture Notes 5

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June 29, 2007

1 Vector Space Axioms

Let V be a set on which addition and scalar multiplication are defined. Suppose all of the following conditions hold for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars r and s.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
3. There exists an element $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.
4. For every $\mathbf{u} \in V$, there exists $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$.
6. $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$.
7. $(rs)\mathbf{u} = r(s\mathbf{u})$.
8. $1 \cdot \mathbf{u} = \mathbf{u}$.

Then we say that V is a vector space. If the field of scalars is $\mathbb{R}$, V is an $\mathbb{R}$-vector space. If the field of scalars is $\mathbb{C}$, then V is a $\mathbb{C}$-vector space. Every $\mathbb{C}$-vector space is also an $\mathbb{R}$-vector space.

2 Vector Space Examples

- $\mathbb{R}^n$ is an $\mathbb{R}$-vector space. $\mathbb{C}^n$ is a $\mathbb{C}$-vector space.
- Matrices, M_{mn}. The above axioms are easy to check. We can write down different generating sets for M_{22}. We can also take linear combinations and spans inside M_{22}.
- Spans. This includes lines, planes, and the zero vector space.
- Polynomials, P_n.
- Continuous functions, $C[a, b]$.

3 Vector Space Properties

- $0 \cdot \mathbf{u} = \mathbf{0}$. Here $\mathbf{0}$ means the additive identity.
- $r \cdot \mathbf{0} = \mathbf{0}$.
- $(-1)\mathbf{u} = -\mathbf{u}$. The thing on the left is scalar multiplication; the thing on the right is the additive inverse of $\mathbf{u}$.
- If $r\mathbf{u} = \mathbf{0}$, then either $r = 0$ or $\mathbf{u} = \mathbf{0}$.