MATH 54 Lecture Notes 3

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1 Inverses and Elementary Matrices

If A is an $n \times n$ matrix, then the following are equivalent:

- A is invertible.
- $AX = B$ has a unique solution for any B.
- $AX = 0$ has only the trivial solution $X = 0$.
- A is row equivalent to I_n.
- A is a product of elementary matrices.

All elementary matrices are invertible, and their inverses are also elementary matrices. Their inverses are obtained by taking the elementary matrices which correspond to the opposite elementary row operations. We can compute inverses of all the elementary matrices from the previous lecture notes in this manner.

Any invertible matrix can be expressed as a product of elementary as matrices as follows. First, write A^{-1} as a product of elementary matrices:

$$A^{-1} = E_n E_{n-1} \cdots E_1.$$

That is, the row operations required to transform A into I are performed by $E_1, E_2, \ldots, E_n$. Then

$$A = (A^{-1})^{-1} = E_1^{-1} E_2^{-1} \cdots E_n^{-1}.$$

Since the inverse of any elementary matrix is also an elementary matrix, this gives A as the product of elementary matrices. We can use this technique to write the invertible matrix from the previous lecture notes as the product of elementary matrices.

A noninvertible matrix cannot be written as the product of elementary matrices, since any product of invertible matrices is invertible. (This proves the equivalence of the first and last conditions above.)

2 Transposes and Symmetric Matrices

Let A be an $n \times n$ matrix, where a_{ij} is the entry in the i-th row and j-th column. Then the *transpose* of A, written A^T, is the $n \times n$ matrix where a_{ji} is the entry in the i-th row and j-th column. Sometimes A^T and A will have different dimensions, such as if $A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \\ 4 & 1 \end{pmatrix}$. However, if A is square, then A and A^T have the same dimensions.

Some facts about transposes:

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- A is invertible if and only if A^T is. If they are both invertible, then $(A^T)^{-1} = (A^{-1})^T$.

A matrix A is called *symmetric* if $A^T = A$. It is called *skew-symmetric* if $A^T = -A$. Any symmetric or skew-symmetric matrix must necessarily be square.