MATH 54 Lecture Notes 26

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1 Fourier Series

Recall the convolution integral inner product on $C[-L,L]$:

$$\langle f,g\rangle = \int_{-L}^{L} f(x)\overline{g(x)}\,dx.$$

Suppose we want to approximate a function $f \in C[-L,L]$ with a linear combination of the functions

$$1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \cos\left(\frac{3\pi x}{L}\right), \sin\left(\frac{3\pi x}{L}\right), \ldots$$

We can do this by taking an orthgonal projection, but first we need the list of functions we are projecting onto to be orthogonal. Let $m > n \geq 0$. Then

$$\begin{aligned}\left\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right\rangle &= \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx \\ &= \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(m+n)\pi x}{L}\right) + \cos\left(\frac{(m-n)\pi x}{L}\right)\,dx \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\left\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right)\right\rangle &= \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx \\ &= \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\,dx \\ &= 0\end{aligned}$$

since both $m+n$ and $m-n$ are positive integers. Now suppose $m \geq n \geq 0$. Then

$$\begin{aligned}\left\langle \cos\left(\frac{m\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\rangle &= \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{1}{2}\int_{-L}^{L} \sin\left(\frac{(m+n)\pi x}{L}\right) - \sin\left(\frac{(m-n)\pi x}{L}\right)\,dx \\ &= 0\end{aligned}$$

since $m+n$ and $m-n$ are both integers. Therefore these vectors are already orthogonal, so the projection of f is given by

$$\frac{\langle f,1\rangle}{\langle 1,1\rangle}\cdot 1 + \sum_{n=1}^{\infty}\left[\frac{\langle f,\cos\left(\frac{n\pi x}{L}\right)\rangle}{\langle \cos\left(\frac{n\pi x}{L}\right),\cos\left(\frac{n\pi x}{L}\right)\rangle}\cdot\cos\left(\frac{n\pi x}{L}\right) + \frac{\langle f,\sin\left(\frac{n\pi x}{L}\right)\rangle}{\langle \sin\left(\frac{n\pi x}{L}\right),\sin\left(\frac{n\pi x}{L}\right)\rangle}\cdot\sin\left(\frac{n\pi x}{L}\right)\right].$$

We will refer to the quantity $\frac{2\langle f,1\rangle}{\langle 1,1\rangle}$ as a_0. Then

$$a_0 = \frac{2\int_{-L}^{L} f(x)\,dx}{\int_{-L}^{L} dx} = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{0\cdot\pi x}{L}\right)\,dx.$$

For $n > 0$, we will refer to the quantity $\frac{\langle f,\cos\left(\frac{n\pi x}{L}\right)\rangle}{\langle \cos\left(\frac{n\pi x}{L}\right),\cos\left(\frac{n\pi x}{L}\right)\rangle}$ as a_n. Then since

$$\left\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right\rangle = \int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right)\,dx = \int_{-L}^{L} \frac{1+\cos\left(\frac{2n\pi x}{L}\right)}{2}\,dx = L,$$

we have that

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$

Note that this formula for a_n coincides with the formula for a_0 if we set $n = 0$.

For $n > 0$, we will refer to the quantity $\frac{\langle f,\sin\left(\frac{n\pi x}{L}\right)\rangle}{\langle \sin\left(\frac{n\pi x}{L}\right),\sin\left(\frac{n\pi x}{L}\right)\rangle}$ as b_n. Since

$$\left\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\rangle = \int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right)\,dx = \int_{-L}^{L} \frac{1-\cos\left(\frac{2n\pi x}{L}\right)}{2}\,dx = L$$

we have that

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Then the *Fourier series* for f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right].$$

Exercise 10.2.14. Let

$$f(x) = \begin{cases} 1, & -L \le x < 0 \\ 0, & 0 \le x < L \end{cases}$$

for $x \in [-L, L)$, and extend f outside this interval by $f(x+2L) = f(x)$ for all $x \in \mathbb{R}$. Then we can use the above formulas to compute the Fourier series of f. For any $n \ge 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{1}{L}\left[\int_{-L}^{0} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx + \int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx\right] \\ &= \frac{1}{L}\int_{-L}^{0} \cos\left(\frac{n\pi x}{L}\right)\,dx. \end{aligned}$$

Then $a_0 = \frac{1}{L}\int_{-L}^{0} dx = 1$, and for any $n > 0$,

$$a_n = \frac{1}{L}\cdot\frac{L}{n\pi}\left(\sin(0) + \sin(n\pi)\right) = 0.$$

If $n > 0$,

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx \\ &= \frac{1}{L} \left[\int_{-L}^{0} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx + \int_{-0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx \right] \\ &= \frac{1}{L} \int_{-L}^{0} \sin\left(\frac{n\pi x}{L}\right) \, dx \\ &= \frac{1}{L} \cdot \frac{L}{n\pi} \left(-\cos(0) + \cos(n\pi)\right) \\ &= -\frac{1-(-1)^n}{n\pi}. \end{aligned}$$

Therefore the Fourier series for f is given by

$$\frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2} - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi x}{L}\right).$$

2 Sine and Cosine Series

2.1 Cosine Series

Suppose $f \in C[-L, L]$ is an even function. Then $f(x) \cos\left(\frac{n\pi x}{L}\right)$ is even for all $n \geq 0$, so that

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx = \frac{2}{L} \int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx.$$

Since $f(x) \sin\left(\frac{n\pi x}{L}\right)$ is odd for all $n > 0$,

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) = 0.$$

If we start with a function $[0, L] \to \mathbb{R}$, we can take the Fourier series of its even extension to $[-L, L]$. This is called the cosine series of the function, since all the sine terms end up having zero as coefficients.

Example. Let $f(x) = x$. Find the cosine series of f with period 2π.

2.2 Sine Series

Suppose $f \in C[-L, L]$ is an odd function. Then $f(x) \sin\left(\frac{n\pi x}{L}\right)$ is even for all $n > 0$, so that

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx.$$

Since $f(x) \cos\left(\frac{n\pi x}{L}\right)$ is odd for all $n \geq 0$,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) = 0.$$

If we start with a function $[0, L] \to \mathbb{R}$, we can take the Fourier series of its odd extension to $[-L, L]$. This is called the sine series of the function, since all the cosine terms end up having zero as coefficients.

Example. Let $f(x) = 1$. Find the cosine series of f with period 2π.