MATH 54 Lecture Notes 25

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1 Boundary Value Problems

In an initial value problem, we are given a linear differential equation of order n for some n, and the values of the function and its derivatives of order less than n at a particular point t_0. In order for the existence and uniqueness theorem to hold, all these initial values must be at the same point t_0. When they are not at the same point, it is called a *boundary value problem*. The existence and uniqueness theorem no longer applies; such problems can have infinitely many, one, or no solutions.

Exercise 10.1.2. Consider the boundary value problem

$$\begin{aligned} y'' + 2y &= 0 \\ y'(0) &= 1 \\ y'(\pi) &= 0. \end{aligned}$$

We first find the general solution of the differential equation as in the case of an initial value problem. In this case, the characteristic equation is $\lambda^2 + 2 = 0$, which has roots $\pm i\sqrt{2}$. Therefore the general solution has the form

$$y = c_1 \cos(t\sqrt{2}) + c_2 \sin(t\sqrt{2}).$$

Then

$$y' = -c_1\sqrt{2}\sin(t\sqrt{2}) + c_2\sqrt{2}\cos(t\sqrt{2}).$$

Since $y'(0) = 1$, $c_2 = \frac{1}{\sqrt{2}}$. Then since $y'(\pi) = 0$,

$$0 = -c_1\sqrt{2}\sin(\pi\sqrt{2}) + \cos(\pi\sqrt{2})$$

so that

$$c_1 = \frac{\cos(\pi\sqrt{2})}{\sqrt{2}\sin(\pi\sqrt{2})} = \frac{1}{\sqrt{2}\tan(\pi\sqrt{2})}.$$

Therefore there is a unique solution,

$$y = \frac{1}{\sqrt{2}}\left[\frac{\cos(t\sqrt{2})}{\tan(\pi\sqrt{2})} + \sin(t\sqrt{2})\right].$$

2 Eigenvalues and Eigenfunctions for Boundary Value Problems

A homogeneous boundary value problem always has the zero function as a solution. If such a problem is parametrized by λ, we can ask for which values of λ does the problem also have nontrivial solutions. These values of λ are called the eigenvalues of the boundary value problem, and the corresponding nontrivial solutions are called eigenfunctions.

Exercise 10.1.16. Consider the boundary value problem

$$\begin{aligned} y'' + \lambda y &= 0 \\ y'(0) &= 0 \\ y'(\pi) &= 0. \end{aligned}$$

For which values of λ are there nontrivial solutions? First we need to find the general solution of the differential equation. To simplify notation, let μ be such that $\mu^2 = -\lambda$. If $\lambda > 0$, then μ is imaginary. The case $\mu = 0$ is special, so we'll deal with it first. In this case the general solution is

$$y = c_1 + c_2 t.$$

Since $y'(0) = y'(\pi) = 0$ and $y' = c_2$, $c_2 = 0$. Therefore all constant functions are solutions when $\lambda = 0$.

Now suppose $\mu \neq 0$. Then the general solution is

$$y = c_1 e^{\mu t} + c_2 e^{-\mu t}$$

and

$$y' = \mu(c_1 e^{\mu t} - c_2 e^{-\mu t}).$$

Since $y'(0)$ must be 0, we have that

$$0 = \mu(c_1 - c_2).$$

Since we are dealing with the case $\mu \neq 0$, we now know that $c_1 = c_2$. Since c_1 and c_2 are the same, we can simply call both of them c. Then

$$y = c(e^{\mu t} + e^{-\mu t})$$

and

$$y' = \mu c(e^{\mu t} - e^{-\mu t}).$$

Then since $y'(\pi) = 0$,

$$0 = \mu c(e^{\mu \pi} - e^{-\mu \pi}).$$

We are assuming here that $\mu \neq 0$. We can also assume that $c \neq 0$, since $c = 0$ would give us the trivial solution, which is not what we're looking for. Therefore

$$e^{\mu \pi} = e^{-\mu \pi}$$

or

$$e^{2\mu\pi} = 1.$$

Then $2\mu\pi$ is a complex number of the form $a + bi$, where

$$e^a = |e^{a+bi}| = |e^{2\mu\pi}| = 1$$

so that $a = 0$. Then

$$1 = e^{bi} = \cos b + i \sin b.$$

Therefore $\cos b = 1$ and $\sin b = 0$. Thus $b = 2\pi n$ for some $n \in \mathbb{Z}$. Then $2\mu\pi = 2\pi n i$, so that $\mu = ni$. Then

$$\lambda_n = -\mu^2 = n^2$$

and

$$y_n = \frac{e^{nit} + e^{-nit}}{2} = \cos(nt)$$

for $n > 0$. To include the case of constant functions from above, we can say this formula applies for all $n \geq 0$.