MATH 54 Lecture Notes 23

GSI Carter

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1 Initial Conditions

Consider the following initial value problem

$$\begin{aligned} \mathbf{x}'(t) &= A\mathbf{x}(t) \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \end{aligned}$$

where $t_0 \neq 0$. Let $\mathbf{y}(t) = \mathbf{x}(t+t_0)$. Then $\mathbf{y}(t-t_0) = \mathbf{x}(t)$, so $\mathbf{y}$ is a solution to the initial value problem

$$\begin{aligned} \mathbf{y}'(t-t_0) &= A\mathbf{y}(t-t_0) \\ \mathbf{y}(0) &= \mathbf{x}_0. \end{aligned}$$

So, when dealing with systems of differential equations of this form, we will assume that $t_0 = 0$.

As an example, consider the following initial value problem.

$$\begin{aligned} y'' + 6y' + 9y &= 0 \\ y(4) &= 1 \\ y'(4) &= 2 \end{aligned}$$

Then if we set $z(t-4) = y(t)$, then z is a solution to

$$\begin{aligned} z'' + 6z' + 9z &= 0 \\ z(0) &= 1 \\ z'(0) &= 2. \end{aligned}$$

The solution is $z = e^{-3t} + 5te^{-3t}$. Therefore the solution to our original initial value problem is $y = e^{-3(t-4)} + 5te^{-3(t-4)}$.

2 Matrix Exponentiation

2.1 Definition

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$. Then we define the exponential of A, denoted $\exp(A)$, to be the matrix

$$\sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2} + \frac{A^3}{6} + \frac{A^4}{24} + \cdots$$

This matrix will play the same role in solving homogeneous systems of first-order linear differential equations that the scalar exponential function serves in solving homogeneous higher-order linear differential equations. First we need to show some properties of the exponential of a matrix.

2.2 Basic Properties

The first properties we would like to show are ones that indicate the the matrix exponential behaves in a similar way to the scalar exponential function. For instance,

$$\exp(0) = I + 0 + \frac{0^2}{2} + \cdots = I.$$

So the exponential of the zero matrix is the identity matrix. Next, suppose A and B are two $n \times n$ matrices such that $AB = BA$. For any two such matrices, the binomial theorem holds. That is,

$$(A+B)^n = \sum_{j=0}^{n} \binom{n}{j} A^j B^{n-j}$$

where

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}.$$

Then

$$\begin{aligned}
\exp(A+B) &= \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} A^j B^{n-j} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{A^j}{j!} \cdot \frac{B^{n-j}}{(n-j)!} \\
&= \exp(A)\exp(B).
\end{aligned}$$

Since A always commutes with $-A$,

$$\exp(A)\exp(-A) = \exp(A - A) = \exp(0) = I.$$

Therefore the exponential of any matrix is always invertible, and $\exp(A)^{-1} = \exp(-A)$. Also, for any matrix A,

$$\begin{aligned}
\frac{d}{dt}\exp(At) &= \frac{d}{dt} \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{n A^n t^{n-1}}{n!} \\
&= \sum_{n=1}^{\infty} \frac{A^n t^{n-1}}{(n-1)!} \\
&= A \sum_{n=1}^{\infty} \frac{A^{n-1} t^{n-1}}{(n-1)!} \\
&= A \exp(At).
\end{aligned}$$

2.3 Exponentiation and Similarity

Now suppose $A = P^{-1}BP$, for some $n \times n$ matrices A, B, and P. We wish to relate $\exp(A)$, $\exp(B)$, and P. We have that

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{P^{-1}B^n P}{n!} = P^{-1} \left(\sum_{n=0}^{\infty} \frac{B^n}{n!} \right) P = P^{-1}\exp(B)P.$$

Therefore, if $A = SJS^{-1}$ where J is a block diagonal matrix (such as the Jordan canonical form of A), we can compute $\exp(A)$ using the formula

$$\exp(A) = S\exp(J)S^{-1}.$$

If

$$J = \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_r \end{pmatrix}$$

for some square matrices $B_1, B_2, \ldots, B_r$, then

$$J^n = \begin{pmatrix} B_1^n & & & \\ & B_2^n & & \\ & & \ddots & \\ & & & B_r^n \end{pmatrix}$$

so that

$$\begin{aligned} \exp(J) &= \sum_{n=0}^{\infty} \frac{J^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} B_1^n & & & \\ & B_2^n & & \\ & & \ddots & \\ & & & B_r^n \end{pmatrix} \\ &= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{B_1^n}{n!} & & & \\ & \sum_{n=0}^{\infty} \frac{B_2^n}{n!} & & \\ & & \ddots & \\ & & & \sum_{n=0}^{\infty} \frac{B_r^n}{n!} \end{pmatrix} \\ &= \begin{pmatrix} \exp(B_1) & & & \\ & \exp(B_2) & & \\ & & \ddots & \\ & & & \exp(B_r) \end{pmatrix}. \end{aligned}$$

So, if we want to calculate $\exp(At)$, it is enough to be able to calculate $\exp(Bt)$ where B is some Jordan block.

2.4 Systems of Homogeneous First-Order Linear Differential Equations with Constant Coefficients

Consider the following initial value problem, where A is some $n \times n$ matrix with constant coefficients.

$$\begin{aligned} \mathbf{x}'(t) &= A\mathbf{x}(t) \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned}$$

The solution to this initial value problem is

$$\mathbf{x}(t) = \exp(At)\mathbf{x}_0.$$

This is a solution to the system of differential equations because $\frac{d}{dt}\exp(At) = A\exp(At)$, and it has the appropriate value at $t = 0$ because $\exp(0) = I$.

Boyce-DiPrima refers to the matrix $\exp(At)$ as the fundamental matrix of this system of differential equations.

2.5 Computation of $\exp(At)$

2.5.1 The exponential of a scalar multiple of the identity

In this case,

$$\exp(\lambda I t) = \sum_{n=0}^{\infty} \frac{\lambda^n I t^n}{n!} = \left(\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \right) I = e^{\lambda t} I. \tag{1}$$

Since matrix exponentials can be computed block-by-block, this already enables us to compute the exponential of any diagonalizable matrix.

Example. We will find the general solution to the differential equation:

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}(t). \tag{2}$$

Call this 2×2 matrix A. Then

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1).$$

The eigenspace for $\lambda = 3$ is

$$NS(A - 3I) = NS \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\},$$

and the eigenspace for $\lambda = -1$ is

$$NS(A + I) = NS \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \text{Span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Therefore, if $S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$, then $A = S \Lambda S^{-1}$. Then

$$\exp(At) = S \exp(\Lambda t) S^{-1} = S \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} S^{-1}.$$

The set of solutions of the equation (2) is equal to the column space of $\exp(At)$. However, $\exp(At)$ has the same column space as

$$\exp(At) S = S \exp(\Lambda t) = \begin{pmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{pmatrix},$$

so the general solution is

$$\mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

2.5.2 Diagonalizable systems of differential equations

The pattern in the previous example generalizes. Let A be an $n \times n$ diagonalizable matrix. Then A has linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $A = S \Lambda S^{-1}$, where $S = (\mathbf{v}_1 \mid \cdots \mid \mathbf{v}_n)$ and Λ is the diagonal matrix with diagonal entires $\lambda_1, \ldots, \lambda_n$. Then $\exp(At) = S \exp(\Lambda t) S^{-1}$, which has the same column space as

$$\exp(At) S = S \exp(\Lambda t) = \left(e^{\lambda_1 t} \mathbf{v}_1 \mid e^{\lambda_2 t} \mathbf{v}_2 \mid \cdots \mid e^{\lambda_n t} \mathbf{v}_n \right).$$

Therefore the general solution to the differential equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 + \cdots + c_n e^{\lambda_n t} \mathbf{v}_n.$$

2.5.3 Diagonalizable systems with complex eigenvalues

Let A be a real 3×3 matrix with eigenvalues λ_0, λ_1, and $\overline{\lambda_1}$, where $\lambda_1 \notin \mathbb{R}$. Then A is diagonalizable, since its eigenvalues are all distinct. Let $\mathbf{v}_0$ be a nonzero eigenvector with eigenvalue λ_0, and let $\mathbf{v}_1$ be a nonzero eigenvector with eigenvalue λ_1. Then

$$A\overline{\mathbf{v}_1} = \overline{A\mathbf{v}_1} = \overline{\lambda_1\mathbf{v}_1} = \overline{\lambda_1}\,\overline{\mathbf{v}_1}.$$

Therefore $\overline{\mathbf{v}_1}$ is a nonzero eigenvector with eigenvalue $\overline{\lambda_1}$. So the general solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$$\begin{aligned}
\mathbf{x}(t) &= c_0 e^{\lambda_0 t}\mathbf{v}_0 + c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\overline{\lambda_1} t}\overline{\mathbf{v}_1} \\
&= c_0 e^{\lambda_0 t}\mathbf{v}_0 + c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 \left(\overline{e^{\lambda_1 t}\mathbf{v}_1}\right) \\
&= c_0 e^{\lambda_0 t}\mathbf{v}_0 + (c_1 + c_2)\mathrm{Re}\left(e^{\lambda_1 t}\mathbf{v}_1\right) + (c_1 - c_2)\mathrm{Im}\left(e^{\lambda_1 t}\mathbf{v}_1\right).
\end{aligned}$$

Therefore our three linearly independent solutions are $e^{\lambda_0 t}\mathbf{v}_0$, $\mathrm{Re}\left(e^{\lambda_1 t}\mathbf{v}_1\right)$, and $\mathrm{Im}\left(e^{\lambda_1 t}\mathbf{v}_1\right)$.

Example. Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

and suppose we want the general solution for $\mathbf{x}'(t) = A\mathbf{x}(t)$. Then

$$\chi_A(\lambda) = \begin{vmatrix} \lambda-1 & -2 & 0 \\ 2 & \lambda-1 & 0 \\ 0 & 0 & \lambda+2 \end{vmatrix} = (\lambda+2)\left[(\lambda-1)^2+4\right] = (\lambda+2)(\lambda-1-2i)(\lambda-1+2i).$$

The eigenspace for $\lambda = -2$ is

$$NS(A+2I) = NS\begin{pmatrix} 3 & 2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\},$$

so our first solution is $e^{-2t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The eigenspace for $\lambda = 1+2i$ is

$$NS(A-I-2iI) = NS\begin{pmatrix} -2i & 2 & 0 \\ -2 & -2i & 0 \\ 0 & 0 & -3-2i \end{pmatrix} = NS\begin{pmatrix} -i & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}\right\}.$$

Then

$$e^{(1+2i)t}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = e^t\left[\cos(2t) + i\sin(2t)\right]\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = e^t\left[\begin{pmatrix} \cos(2t) \\ -\sin(2t) \\ 0 \end{pmatrix} + i\begin{pmatrix} \sin(2t) \\ \cos(2t) \\ 0 \end{pmatrix}\right].$$

Therefore the general solution is

$$\mathbf{x}(t) = c_1 e^{-2t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + e^t\left[c_2\begin{pmatrix} \cos(2t) \\ -\sin(2t) \\ 0 \end{pmatrix} + c_3\begin{pmatrix} \sin(2t) \\ \cos(2t) \\ 0 \end{pmatrix}\right].$$

2.5.4 The exponential of a 2×2 Jordan block

Define a matrix

$$N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then any 2×2 Jordan block B is equal to $\lambda I + N_2$ for some λ. Note that λIt and $N_2 t$ commute, since the former is a scalar multiple of I, so that $\exp(Bt) = \exp(\lambda It + N_2 t) = \exp(\lambda It)\exp(N_2 t)$. We already computed $\exp(\lambda It) = e^{\lambda t} I$ in (1), so we only need to compute $\exp(N_2 t)$. Since $N_2^2 = 0$,

$$\exp(N_2 t) = \sum_{n=0}^{\infty} \frac{N_2^n t^n}{n!} = \sum_{n=0}^{1} \frac{N_2^n t^n}{n!} = I + N_2 t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

This gives us that

$$\exp(Bt) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

Example 1. Consider the differential equation

$$\mathbf{x}'(t) = \frac{1}{3} \begin{pmatrix} 2 & -4 & 5 \\ 1 & 1 & 1 \\ 3 & 0 & 0 \end{pmatrix} \mathbf{x}(t). \tag{3}$$

Then if

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

and

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

then SJS^{-1} is the matrix from the equation (3). Then the general solution is an element of the column space of

$$S\exp(Jt) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^t & te^t + e^t & e^{-t} \\ e^t & te^t - e^t & 0 \\ e^t & te^t & -e^{-t} \end{pmatrix},$$

so the general solution to (3) is

$$\mathbf{x}(t) = e^t \left[c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} t+1 \\ t-1 \\ t \end{pmatrix} \right] + c_3 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Example 2. Consider the differential equation

$$\mathbf{x}'(t) = \frac{1}{3} \begin{pmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{pmatrix} \mathbf{x}(t). \tag{4}$$

Then if

$$S = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

and

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then SJS^{-1} is the matrix from the equation (4). Then the general solution is an element of the column space of

$$S\exp(Jt) = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} = \begin{pmatrix} e^t & te^t + 3e^t & e^t \\ e^t & te^t & 0 \\ e^t & te^t & -e^t \end{pmatrix},$$

so the general solution to (4) is

$$\mathbf{x}(t) = e^t \left[c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} t+3 \\ t \\ t \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right].$$

2.5.5 The exponential of a 3×3 Jordan block

Define a matrix

$$N_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

As above, any 3×3 Jordan block B is equal to $\lambda I + N_3$ for some λ. Also, λIt and $N_3 t$ commute, so that $\exp(Bt) = \exp(\lambda It + N_3 t) = \exp(\lambda It)\exp(N_3 t) = e^{\lambda t}\exp(N_3 t)$ by (1). Since $N_3^3 = 0$,

$$\exp(N_3 t) = \sum_{n=0}^{2} \frac{N_3^n t^n}{n!} = I + N_3 t + \frac{N_3^2 t^2}{2} = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$\exp(Bt) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2 e^{\lambda t}}{2} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

Example. Consider the differential equation

$$\mathbf{x}'(t) = \frac{1}{3} \begin{pmatrix} 5 & -1 & -1 \\ 0 & 0 & 3 \\ 1 & -2 & 4 \end{pmatrix} \mathbf{x}(t). \tag{5}$$

Then if

$$S = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

and

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

then SJS^{-1} is the matrix from the equation (5). Then the general solution is an element of the column space of

$$S\exp(Jt) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^t & te^t & \frac{t^2 e^t}{2} \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix} = \begin{pmatrix} e^t & te^t + 2e^t & \frac{t^2 e^t}{2} + 2te^t + 3e^t \\ e^t & te^t & \frac{t^2 e^t}{2} \\ e^t & te^t + e^t & \frac{t^2 e^t}{2} + te^t \end{pmatrix},$$

so the general solution to (5) is

$$\mathbf{x}(t) = e^t \left[c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} t+2 \\ t \\ t+1 \end{pmatrix} + c_3 \begin{pmatrix} t^2+4t+6 \\ t^2 \\ t^2+2t \end{pmatrix} \right].$$