MATH 54 Lecture Notes 21

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August 2, 2007

1 Existence and Uniqueness Theorem

Consider the differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = g(t) \tag{1}$$

where each $p_i(t)$ and $g(t)$ are all continuous on an open interval I containing a point t_0. Suppose also we are given the following initial conditions.

$$\begin{aligned} y(t_0) &= c_0 \\ y'(t_0) &= c_1 \\ &\vdots \\ y^{(n-1)}(t_0) &= c_{n-1} \end{aligned}$$

Then there exists a unique solution to (1) satisfying the given initial conditions. This is Theorem 4.1.1 in Boyce-DiPrima.

We can interpret this theorem in terms of what we already know about linear algebra. In doing so, we will consider the equation (1) without the given initial conditions.

First, we need to reduce to the case of a homogeneous linear differential equation, so that the set of solutions is a linear vector space. If $y_1(t)$ and $y_2(t)$ are two solutions to (1), then $y_1 - y_2$ is a solution to the differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0 \tag{2}$$

which is homogeneous, and therefore its solution set is a subspace. Therefore, if we denote the set of solutions of (2) by S, then the set of solutions of (1) is the set

$$\{y_p + y \mid y \in S\}$$

for some fixed solution y_p of (1). This is why we can, for the most part, restrict our attention to homogeneous differential equations.

Define a linear transformation $T : S \to \mathbb{C}^n$ as follows.

$$T(f) = \begin{pmatrix} f(t_0) \\ f'(t_0) \\ \vdots \\ f^{(n-1)}(t_0) \end{pmatrix}$$

The existence part of the theorem tells us that T is onto. That is, for any $\mathbf{v} \in \mathbb{C}^n$, there exists $f \in S$ such that $T(f) = \mathbf{v}$. It is a general fact of linear algebra that if $T(f_1), T(f_2), \ldots, T(f_r)$ are linearly independent, then so are $f_1, f_2, \ldots, f_r$. Now suppose $f_1, f_2, \ldots, f_r$ are linearly independent, and suppose

$$a_1 T(f_1) + a_2 T(f_2) + \cdots + a_r T(f_r) = \mathbf{0}.$$

Then since T is a linear transformation,

$$T(a_1 f_1 + a_2 f_2 + \cdots + a_r f_r) = \mathbf{0}.$$

Now we'll apply the uniqueness part of the theorem. With initial conditions

$$y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0,$$

the zero function is a solution to (2). By uniqueness, it is the only solution. Therefore,

$$a_1 f_1 + a_2 f_2 + \cdots + a_r f_r = 0.$$

Since the f_i's are linearly independent, this implies that

$$a_1 = a_2 = \cdots = a_r = 0.$$

Therefore $T(f_1), T(f_2), \ldots, T(f_r)$ is a linearly independent set of vectors in $\mathbb{C}^n$.

Let $f_1, f_2, \ldots, f_m$ be a basis for S. Then $T(f_1), T(f_2), \ldots, T(f_m)$ is a linearly independent set of vectors, and they span $\mathbb{C}^n$ because T is onto. Therefore they form a basis for $\mathbb{C}^n$. This tells us that $m = n$, so S is n-dimensional.

Suppose we have $f_1, f_2, \ldots, f_n \in S$, and we want to know whether they're linearly independent (and therefore a basis for S). Then we only have to test whether $T(f_1), T(f_2), \ldots, T(f_n)$ are linearly independent in $\mathbb{C}^n$. This is true if and only if the matrix with each of these vectors as columns is invertible, which in turn is equivalent to

$$\begin{vmatrix} f_1(t_0) & f_2(t_0) & \cdots & f_n(t_0) \\ f_1'(t_0) & f_2'(t_0) & \cdots & f_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(t_0) & f_2^{(n-1)}(t_0) & \cdots & f_n^{(n-1)}(t_0) \end{vmatrix} \neq 0. \tag{3}$$

2 The Wronskian

The *Wronskian* of $f_1, f_2, \ldots, f_n$, denoted $W(f_1, f_2, \ldots, f_n)$, is the determinant

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

This determinant is a function of t. Here we are thinking of these functions as scalars that vary with t, not as vectors. If $W(f_1, f_2, \dots, f_n)(t_0) \neq 0$ for some t_0, then $f_1, f_2, \dots, f_n$ are linearly independent. If $f_1, f_2, \dots, f_n$ are solutions to an n-th order homogeneous linear differential equation such as (2), then we can rephrase (3) as

$$W(f_1, f_2, \dots, f_n)(t_0) \neq 0.$$

In particular, if the Wronskian is zero at t_0, $f_1, f_2, \dots, f_n$ are linearly dependent. This implication only holds if the functions are solutions to (2).

We can use the Wronskian to verify that our general solution formulas from last time did actually give all solutions to the corresponding differential equations.

In all cases, we are trying to solve the differential equation

$$y'' + ay' + by = 0. \tag{4}$$

Case 1: $\lambda^2 + a\lambda + b = (\lambda - r_1)(\lambda - r_2)$ where $r_1 \neq r_2$ and $r_1, r_2 \in \mathbb{R}$. Here, our two solutions are $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$. Then

$$W(y_1, y_2) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)e^{(r_1 + r_2)t}.$$

This is nonzero for all t, so y_1 and y_2 for a basis for the solution set of (4).

Case 2: $\lambda^2 + a\lambda + b = (\lambda - r)^2$. Now $y_1(t) = e^{rt}$ and $y_2(t) = te^{et}$. Therefore

$$W(y_1, y_2) = \begin{vmatrix} e^{rt} & te^{rt} \\ re^{rt} & (rt+1)e^{rt} \end{vmatrix} = e^{2rt}$$

which is also nonzero for all t. Therefore y_1 and y_2 are a basis for the solution set of (4).

Case 3: $\lambda^2 + a\lambda + b = (\lambda - \alpha - \beta i)(\lambda - \alpha + \beta i)$ where $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. The solutions we ended up with after some computation were $f_1(t) = e^{\alpha t}\cos(\beta t)$ and $f_2(t) = e^{\alpha t}\sin(\beta t)$. We can compute the Wronskian of these solutions directly.

$$W(f_1, f_2) = \begin{vmatrix} e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ e^{\alpha t}(\alpha\cos(\beta t) - \beta\sin(\beta t) & e^{\alpha t}(\alpha\sin(\beta t) + \beta\cos(\beta t)) \end{vmatrix} = \beta e^{2\alpha t}$$

This is nonzero for all t since β is assumed to be nonzero. Therefore f_1 and f_2 are a basis for the solution set.

At first, we had the solutions $g_1(t) = e^{(\alpha+\beta i)t}$ and $g_2(t) = e^{(\alpha-\beta i)t}$. We can use the computation from case 1 to determine $W(g_1, g_2)$ by setting $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$.

$$W(g_1, g_2) = (\alpha - \beta i - \alpha - \beta i)e^{(\alpha+\beta i+\alpha-\beta i)t} = -2\beta i e^{2\alpha t}$$

This is again nonzero for all t, so g_1 and g_2 are a basis for the solution set. We obtained the real-valued solutions f_1 and f_2 by taking $f_1 = \frac{g_1+g_2}{2}$ and $f_2 =$

$\frac{g_1-g_2}{2i}$. We can use these relations to compute $W(f_1, f_2)$ based on $W(g_1, g_2)$. We have that

$$\begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} = \begin{pmatrix} \frac{g_1+g_2}{2} & \frac{g_1-g_2}{2i} \\ \frac{g_1'+g_2'}{2} & \frac{g_1'-g_2'}{2i} \end{pmatrix} = \begin{pmatrix} g_1 & g_2 \\ g_1' & g_2' \end{pmatrix} \begin{pmatrix} 1/2 & 1/2i \\ 1/2 & -1/2i \end{pmatrix}.$$

Therefore

$$W(f_1, f_2) = \det \begin{pmatrix} 1/2 & 1/2i \\ 1/2 & -1/2i \end{pmatrix} W(g_1, g_2) = \left(-\frac{1}{2i}\right)\left(-2\beta i e^{2\alpha t}\right) = \beta e^{2\alpha t}.$$