MATH 54 Lecture Notes 2

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1 Matrix Arithmetic

1.1 Multiplying Matrices and Vectors

Let A be an $m \times n$ matrix with entries a_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq n$, and let $\mathbf{b}$ be an n element column vector with entries b_j for $1 \leq j \leq n$. Then $\mathbf{c} = A\mathbf{b}$ is an m element column vector. If we call the entries of $\mathbf{c}$ by c_i for $1 \leq i \leq m$, then

$$c_i = \sum_{j=1}^{n} a_{ij} b_j. \tag{1}$$

Some examples:

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -9 \\ 1 \end{pmatrix} = \begin{pmatrix} -15 \\ -10 \end{pmatrix}$$

$$\begin{pmatrix} i & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = \begin{pmatrix} 3i \\ 4i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

If the number of columns of the matrix is not equal to the number of entries of the vector, the product is not defined.

1.2 Multiplying Matrices

Let A be an $m \times n$ matrix with entries a_{ij} and B be an $n \times q$ matrix with entires b_{jk}. Then $C = AB$ is an $m \times q$ matrix. If we denote its entries by c_{ij}, where $1 \leq i \leq m$ and $1 \leq j \leq q$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Another way of putting this is to write B as a matrix of column vectors:

$$B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_q \end{pmatrix}$$

where each $\mathbf{b}_k$ is an n element vector, the j-th entry of which is b_{jk}. Then

$$AB = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_q \end{pmatrix}$$

where each $A\mathbf{b}_k$, being a matrix times a column vector, can be computed using (1).

Some examples:

$$\begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & 5 \end{pmatrix} \begin{pmatrix} 4 & 5 & -1 & 3 \\ 1 & -2 & 0 & 1 \\ 2 & -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -1 & 2 \\ -2 & -20 & 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & i \end{pmatrix}$$

$$\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 2 \\ 1 & 0 \end{pmatrix}$$

Note that matrix multiplication is not commutative.

If the number of columns of the first matrix is not equal to the number of rows of the second matrix, then the product is not defined.

We can also multiply matrices by scalars (that is, by elements of $\mathbb{R}$ or $\mathbb{C}$). To do this, simply multiply each entry of the matrix by the given scalar.

1.3 Adding Matrices

Let A and B both be $m \times n$ matrices, with entries a_{ij} and b_{ij}, respectively. Then $C = A + B$ is also an $m \times n$ matrix, with entries

$$c_{ij} = a_{ij} + b_{ij}.$$

Some examples:

$$\begin{pmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 9 & -3 \\ 0 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 12 & -3 \\ 0 & -2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}$$

If the matrices do not have the same dimensions (that is, the same number of rows *and* the same number of columns), then the sum is not defined.

1.4 Some Special Matrices

1.4.1 The Zero Matrix

The matrix $\mathbf{0}$ is a matrix of any dimension with every entry equal to 0. (The dimension should be chosen to make things work.) Multiplying $\mathbf{0}$ by any matrix or vector gives $\mathbf{0}$, except possibly with different dimensions. This matrix is also the additive identity.

1.4.2 The Identity Matrix

For any n, the $n \times n$ identity matrix, written I_n, is the matrix with every diagonal entry equal to 1 and every other entry equal to 0. Any matrix or vector multiplied by I_n, on either the left or the right, will be the same matrix or vector (as long as n is chosen appropriately).

2 Elementary Matrices and Matrix Inverses

2.1 Definition of the Inverse

If A is an $n \times n$ matrix, then A^{-1} is an $n \times n$ matrix such that

$$AA^{-1} = I_n = A^{-1}A.$$

If A has an inverse, then we say that A is *invertible*. It only makes sense to talk about the inverses of square matrices.

Some facts about inverse matrices:

- If a matrix has the correct size and is an inverse of A on the left (or on the right), then that matrix is A^{-1}.
- Matrix inverses are unique, when they exist.
- If A and B are invertible $n \times n$ matrices, then so is AB, and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Inverses are not as well-behaved with respect to addition. In particular, you can add two non-invertible matrices and get an invertible matrix, or add two invertible matrices and get a non-invertible matrix.

2.2 Elementary Matrices

Each of the three types of elementary row operations can be performed by multiplication on the left by a square matrix. These matrices are invertible. For example,

$$M_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -5 & \\ & & & 1 \end{pmatrix}$$

multiplies the third row by -5. For instance,

$$M_3 \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 4 \\ 2 & 0 & 1 \\ 0 & 9 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 4 \\ -10 & 0 & -5 \\ 0 & 9 & 2 \end{pmatrix}.$$

The matrix

$$P_{12} = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}$$

interchanges rows 1 and 2, and the matrix

$$E_{12} = \begin{pmatrix} 1 & & \\ -3 & 1 & \\ & & 1 \end{pmatrix}$$

adds -3 times the first row to the second row. For instance,

$$P_{12} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 \end{pmatrix},$$

and

$$E_{12} \begin{pmatrix} 1 & 0 \\ 3 & -1 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 4 & 5 \end{pmatrix}.$$

In general, let E be the matrix which performs some particular elementary row operation we have in mind. Then when we perform that row operation to the identity matrix I, we should get $EI = E$. Therefore any elementary matrix can be obtained by simply applying the desired elementary row operation to the identity matrix.

2.3 Computing Inverses

Let A be some $n \times n$ matrix. Suppose $E_1, E_2, \ldots, E_r$ are elementary matrices such that $E_1 E_2 \cdots E_r A = I$. Then $A^{-1} = E_1 E_2 \cdots E_r$. Therefore, whatever row operations turn A into I, those same row operations turn I into A^{-1}. Therefore $(A \mid I)$ row reduces to $\left(I \mid A^{-1}\right)$ if A is invertible. When A is not invertible, A will row reduce to a matrix with at least one row of all zeroes.

Consider the following matrices:

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 4 & 2 \end{pmatrix}$$
$$B = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix}$$