MATH 54 Lecture Notes 19

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1 Block Diagonal Matrices

A *block diagonal matrix* is a square matrix of the form

$$\begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_r \end{pmatrix}$$

where each B_i is a square matrix. When we have two block diagonal matrices with the same sequence of block sizes, they are easier to multiply together. Namely, we can multiply block diagonal matrices together the same way we multiply diagonal matrices together, except here we have to worry about blocks that don't necessarily commute.

Example. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then A and B are both block diagonal. In particular, $A = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}$ where $A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $A_2 = 3$, and $B = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix}$ where $B_1 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ and $B_2 = -1$. Note that when we talk about matrices being block diagonal, the blocks should never intersect each other. Then

$$AB = \begin{pmatrix} A_1B_1 & \\ & A_2B_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} B_1A_1 & \\ & B_2A_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

We can also use this approach to raise a block diagonal matrix to a power:

$$\begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_r \end{pmatrix}^n = \begin{pmatrix} B_1^n & 0 & \cdots & 0 \\ 0 & B_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_r^n \end{pmatrix}.$$

In the above example, $A^2 = \begin{pmatrix} A_1^2 & \\ & A_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 9 \end{pmatrix}$. With small enough blocks, raising a block diagonal matrix to a power is not much more difficult than raising a diagonal matrix to a power. Therefore, when a matrix is not diagonalizable, we will settle for finding a block diagonal matrix that it's similar to, and the smaller the blocks are the better.

2 Definition of the Jordan Canonical Form

A *Jordan block* is a square matrix of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

That is, every diagonal entry is λ, for some $\lambda \in \mathbb{C}$, and every entry just above the main diagonal is 1. All other entries are 0. The following 1×1 matrix is also said to be a Jordan block:

$$(\lambda)$$

The *Jordan canonical form* of a square matrix A is a matrix of the form

$$J = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_r \end{pmatrix}$$

where each B_i is a Jordan block, and J is similar to A. We refer to finding S and J such that $A = SJS^{-1}$ as putting A in Jordan canonical form. For any diagonalizable matrix, its Jordan canonical form is its diagonalization. In this case, each Jordan block is of size 1×1.

We will use the following fact from MATH 110 to compute the Jordan canonical form of matrices: if A is an $n \times n$ matrix with characteristic polynomial

$$\chi_A(\lambda) = (\lambda - \lambda_1)^{r_1}(\lambda - \lambda_2)^{r_2} \cdots (\lambda - \lambda_s)^{r_s},$$

then

$$\dim NS(A - \lambda_i I)^{r_i} = r_i$$

for all i.

These facts from MATH 110 are also useful things to know about the Jordan canonical form, although we won't need them in the following computations:

- Every matrix has a Jordan canonical form.
- The Jordan canonical form of a matrix is unique, up to a rearrangement of the Jordan blocks.

3 The Jordan Canonical Form of a 2×2 Matrix

3.1 A 2×2 matrix with two distinct eigenvalues

Any such matrix is diagonalizable. To put such a matrix into Jordan canonical form, simply diagonalize it.

3.2 A 2×2 matrix with a unique eigenvalue

Call the matrix A, and call this eigenvalue λ_0. Then the characteristic polynomial is $\det(\lambda I - A) = (\lambda - \lambda_0)^2$.

3.2.1 $\dim NS(A-\lambda_0 I)=2$

In this case, A is diagonalizable.

3.2.2 $\dim NS(A-\lambda_0 I)=1$.

Then $\dim NS(A-\lambda_0 I)^2=2$. That is, $NS(A-\lambda_0 I)^2=\mathbb{C}^2$. Let $\mathbf{v}_2\in\mathbb{C}^2$ be such that $(A-\lambda_0 I)\mathbf{v}_2\neq\mathbf{0}$, which must exist since $\dim NS(A-\lambda_0 I)<2$. Since $\mathbf{v}_2\in NS(A-\lambda_0 I)^2=\mathbb{C}^2$, $(A-\lambda_0 I)^2\mathbf{v}_2=\mathbf{0}$. If we let $\mathbf{v}_1=(A-\lambda_0 I)\mathbf{v}_2$, then $\mathbf{v}_1$ is an eigenvector of A, since

$$(A-\lambda_0 I)\mathbf{v}_1=(A-\lambda_0 I)^2\mathbf{v}_2=\mathbf{0}.$$

Then we have that

$$\begin{aligned} A\mathbf{v}_1&=\lambda_0\mathbf{v}_1+0\cdot\mathbf{v}_2\\ A\mathbf{v}_2&=\mathbf{v}_1+\lambda_0\mathbf{v}_2. \end{aligned}$$

Since $\mathbf{v}_2\notin NS(A-\lambda_0 I)$, $\mathbf{v}_1\neq\mathbf{0}$. Therefore the set $\{\mathbf{v}_1\}$ is linearly independent. Since $\mathbf{v}_1\in NS(A-\lambda_0 I)$ and $\mathbf{v}_2\notin NS(A-\lambda_0 I)$, $\mathbf{v}_2\notin\operatorname{Span}\{\mathbf{v}_1\}$. Therefore the set $\{\mathbf{v}_1,\mathbf{v}_2\}$ is linearly independent, and is hence a basis for $\mathbb{C}^2$. By the above equations, A relative to the ordered basis $\{\mathbf{v}_1,\mathbf{v}_2\}$ is the matrix

$$J=\begin{pmatrix}\lambda_0 & 1\\ 0 & \lambda_0\end{pmatrix}.$$

The entire matrix J is a single 2×2 Jordan block. So, if S is the matrix with first column $\mathbf{v}_1$ and second column $\mathbf{v}_2$, then $A=SJS^{-1}$.

Example. Let

$$A=\begin{pmatrix}3 & -1\\ 1 & 1\end{pmatrix}.$$

Then the characteristic polynomial of A is

$$\det(\lambda I-A)=\begin{vmatrix}\lambda-3 & 1\\ -1 & \lambda-1\end{vmatrix}=\lambda^2-4\lambda+3+1=(\lambda-2)^2.$$

The eigenspace for $\lambda=2$ is

$$NS(A-2I)=NS\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}=\operatorname{Span}\left\{\begin{pmatrix}1\\ 1\end{pmatrix}\right\}.$$

Now we know that this matrix is not diagonalizable, so we'll put it into Jordan canonical form instead. First let

$$\mathbf{v}_2=\begin{pmatrix}1\\ 0\end{pmatrix}.$$

Then let

$$\mathbf{v}_1=(A-2I)\mathbf{v}_2=\begin{pmatrix}1\\ 1\end{pmatrix}.$$

These are the columns of S, so

$$S=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$$

and

$$J=\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}.$$

Then $A=SJS^{-1}$.

4 The Jordan Canonical Form of a 3×3 Matrix

4.1 With Three Distinct Eigenvalues

Any such matrix is diagonalizable. Therefore, the Jordan canonical form is the diagonalization.

4.2 With Two Distinct Eigenvalues

One of the eigenvalues must be a double root of the characteristic polynomial, and the other eigenvalue is not repeated. Call the matrix A, the repeated eigenvalue λ_0, and the other eigenvalue λ_1. We are assuming here that $\lambda_0 \neq \lambda_1$. Then $\det(\lambda I - A) = (\lambda - \lambda_0)^2(\lambda - \lambda_1)$. Regardless of which matrix A is, $\dim NS(A - \lambda_1 I) = 1$.

4.2.1 $\dim NS(A - \lambda_0 I) = 2$

In this case, A is diagonalizable.

4.2.2 $\dim NS(A - \lambda_0 I) = 1$

Since $\dim NS(A - \lambda_0 I)^2 = 2$, we can choose $\mathbf{v}_2 \in NS(A - \lambda_0 I)^2$ such that $\mathbf{v}_2 \notin NS(A - \lambda_0 I)$. Then let $\mathbf{v}_3$ be some nonzero eigenvector in $NS(A - \lambda_1 I)$ and $\mathbf{v}_1 = (A - \lambda_0 I)\mathbf{v}_2$. Since $\mathbf{v}_2 \in NS(A - \lambda_0 I)^2$, $\mathbf{v}_1 \in NS(A - \lambda_0 I)$. Since $\mathbf{v}_2 \notin NS(A - \lambda_0 I)$, $\mathbf{v}_1 \neq \mathbf{0}$. Therefore the set $\{\mathbf{v}_1\}$ is linearly independent. Then since $\mathbf{v}_1 \in NS(A - \lambda_0 I)$ and $\mathbf{v}_2 \notin NS(A - \lambda_0 I)$, $\mathbf{v}_2 \notin \operatorname{Span}\{\mathbf{v}_1\}$, so the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Since $NS(A - \lambda_0 I) \subseteq NS(A - \lambda_0 I)^2$ (this is a special case of the general theorem which says that $NS(B) \subseteq NS(AB)$), $\mathbf{v}_1, \mathbf{v}_2 \in NS(A - \lambda_0 I)^2$. However,

$$(A - \lambda_0 I)^2 \mathbf{v}_3 = (\lambda_1 - \lambda_0)^2 \mathbf{v}_3 \neq \mathbf{0},$$

so $\mathbf{v}_3 \notin \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Therefore the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{C}^3$, so we can compute the matrix A relative to this basis:

$$\begin{aligned} A\mathbf{v}_1 &= \lambda_0 \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 + 0 \cdot \mathbf{v}_3 \\ A\mathbf{v}_2 &= \mathbf{v}_1 + \lambda_0 \mathbf{v}_2 + 0 \cdot \mathbf{v}_3 \\ A\mathbf{v}_3 &= 0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 + \lambda_1 \mathbf{v}_3. \end{aligned}$$

Thus A relative to this basis is

$$J = \begin{pmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}.$$

This matrix is in Jordan canonical form, with one 2×2 Jordan block and one 1×1 Jordan block. Then if $S = (\mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3)$, $A = SJS^{-1}$.

With this type of matrix, it's often easier to compute $\mathbf{v}_1$ and $\mathbf{v}_2$ by first letting $\mathbf{v}_1$ be some nonzero eigenvector in $NS(A - \lambda_0 I)$ and then solve the matrix equation $(A - \lambda_0 I)\mathbf{v}_2 = \mathbf{v}_1$ for the vector $\mathbf{v}_2$. This way we don't have to compute the matrix $(A - \lambda_0 I)^2$.

Example. Let

$$A = \frac{1}{3} \begin{pmatrix} 2 & -4 & 5 \\ 1 & 1 & 1 \\ 3 & 0 & 0 \end{pmatrix}.$$

Then the characteristic polynomial is

$$\begin{aligned}\det(\lambda I - A) &= \frac{1}{27}\det(3\lambda I - 3A)\\ &= \frac{1}{27}\begin{vmatrix} 3\lambda-2 & 4 & -5\\ -1 & 3\lambda-1 & -1\\ -3 & 0 & 3\lambda\end{vmatrix}\\ &= \frac{1}{27}\left[-3(-4+15\lambda-5)) + 3\lambda(9\lambda^2-9\lambda+2+4)\right]\\ &= \frac{1}{27}\left[27\lambda^3 - 27\lambda^2 - 27\lambda + 27\right]\\ &= \lambda^3-\lambda^2-\lambda+1\\ &= (\lambda-1)^2(\lambda+1).\end{aligned}$$

The eigenspace for $\lambda = -1$ is

$$NS(A+I) = NS(3A+3I) = NS\begin{pmatrix} 5 & -4 & 5\\ 1 & 4 & 1\\ 3 & 0 & 3\end{pmatrix} = NS\begin{pmatrix} 1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 1\\ 0\\ -1\end{pmatrix}\right\}.$$

The eigenspace for $\lambda = 1$ is

$$NS(A-I) = NS(3A-3I) = NS\begin{pmatrix} -1 & -4 & 5\\ 1 & -2 & 1\\ 3 & 0 & -3\end{pmatrix} = NS\begin{pmatrix} 1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 1\\ 1\\ 1\end{pmatrix}\right\}.$$

So the matrix is not diagonalizable. We will let $\mathbf{v}_1 = \begin{pmatrix} 1\\ 1\\ 1\end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 1\\ 0\\ -1\end{pmatrix}$. Then $\mathbf{v}_2$ must satisfy the equation $(A-I)\mathbf{v}_2 = \mathbf{v}_1$. Equivalently, it must satisfy $(3A-3I)\mathbf{v}_2 = 3\mathbf{v}_1$. This matrix equation corresponds to the augmented matrix

$$\begin{pmatrix} -1 & -4 & 5 & 3\\ 1 & -2 & 1 & 3\\ 3 & 0 & -3 & 3\end{pmatrix}$$

which row reduces to

$$\begin{pmatrix} 1 & 0 & -1 & 1\\ 0 & 1 & -1 & -1\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

There is not a unique solution, but any solution will do. Let $\mathbf{v}_2 = \begin{pmatrix} 1\\ -1\\ 0\end{pmatrix}$. Then

$$S = (\mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3) = \begin{pmatrix} 1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 0 & -1\end{pmatrix}$$

and

$$J = \begin{pmatrix} 1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Then $A = SJS^{-1}$.

4.3 With a Unique Eigenvalue

Call the matrix A, and call the eigenvalue λ_0. Then $\det(\lambda I - A) = (\lambda-\lambda_0)^3$ and $\dim NS(A-\lambda_0 I)^3 = 3$, but $\dim NS(A-\lambda_0 I)$ can be either one, two, or three.

4.3.1 $\dim NS(A - \lambda_0 I) = 3$

In this case, A is diagonalizable.

4.3.2 $\dim NS(A - \lambda_0 I) = 2$

Here we will need to know that $\dim NS(A - \lambda_0 I)^2 = 3$. Suppose $\dim NS(A - \lambda_0 I)^2 = 2$. Then $NS(A - \lambda_0 I)^2 = NS(A - \lambda_0 I)$, so $(A - \lambda_0 I)^2 \mathbf{v} = \mathbf{0}$ if and only if $(A - \lambda_0 I)\mathbf{v} = \mathbf{0}$. Now suppose $(A - \lambda_0 I)^3 \mathbf{v} = \mathbf{0}$. Then

$$(A - \lambda_0 I)^2 \left[(A - \lambda_0 I)\mathbf{v}\right] = \mathbf{0}$$

so

$$(A - \lambda_0 I) \left[(A - \lambda_0 I)\mathbf{v}\right] = \mathbf{0}.$$

Therefore $NS(A - \lambda_0 I)^3 \subseteq NS(A - \lambda_0 I)^2$, so $\dim NS(A - \lambda_0 I)^3 = 2$. This is a contradiction, which tells us that $\dim NS(A - \lambda_0 I)^2 = 3$; that is, $NS(A - \lambda_0 I) = \mathbb{C}^3$.

Since $\dim NS(A - \lambda_0 I)^2 > \dim NS(A - \lambda_0 I)$, we can choose $\mathbf{v}_2 \in NS(A - \lambda_0 I)^2$ such that $\mathbf{v}_2 \notin NS(A - \lambda_0 I)$. Then if we let $\mathbf{v}_1 = (A - \lambda_0 I)\mathbf{v}_2$, $\mathbf{v}_1$ is a nonzero eigenvector with eigenvalue λ_0. Then since there is a two-dimensional eigenspace, we can let $\mathbf{v}_3$ be another eigenvector such that $\{\mathbf{v}_1, \mathbf{v}_3\}$ form a basis for $NS(A - \lambda_0 I)$. Then since $\mathbf{v}_2 \notin NS(A - \lambda_0 I)$, $\mathbf{v}_2 \notin \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$, and hence $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{C}^3$. Now we can compute the matrix for A relative to this basis:

$$\begin{aligned} A\mathbf{v}_1 &= \lambda_0 \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 + 0 \cdot \mathbf{v}_3 \\ A\mathbf{v}_2 &= \mathbf{v}_1 + \lambda_0 \mathbf{v}_2 + 0 \cdot \mathbf{v}_3 \\ A\mathbf{v}_3 &= 0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 + \lambda_0 \mathbf{v}_3. \end{aligned}$$

Thus A relative to this basis is

$$J = \begin{pmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix}.$$

If $S = (\mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3)$, then $A = SJS^{-1}$.

Example. Let

$$A = \frac{1}{3} \begin{pmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{pmatrix}.$$

Then the characteristic polynomial is $\det(\lambda I - A) = (\lambda - 1)^3$. Then eigenspace for $\lambda = 1$ is

$$NS(A - I) = NS(3A - 3I) = NS \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Then we can let $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$. Then $\mathbf{v}_1 = (A - I)\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and we can let $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Then

$$S = (\mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3) = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

and

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $A = SJS^{-1}$.

4.3.3 $\dim NS(A-\lambda_0 I) = 1$

Here we need to know that $\dim NS(A-\lambda_0 I)^2 < \dim NS(A-\lambda_0 I)^3$. Suppose otherwise. Then $NS(A-\lambda_0 I)^2 = \mathbb{C}^3$, so that $CS(A-\lambda_0 I) \subseteq NS(A-\lambda_0 I)$. However, we know that $\dim NS(A-\lambda_0 I) = 1$, and the Rank-Nullity Theorem tells us that $\dim CS(A-\lambda_0 I) = 3 - \dim NS(A-\lambda_0 I) = 2$. This is a contradiction.

Since $\dim NS(A-\lambda_0 I)^2 < \dim NS(A-\lambda_0 I)^3$, we can choose $\mathbf{v}_3 \in NS(A-\lambda_0 I)^3 = \mathbb{C}^3$ such that $\mathbf{v}_3 \notin NS(A-\lambda_0 I)^2$. Then we can let $\mathbf{v}_2 = (A-\lambda_0 I)\mathbf{v}_3$ and $\mathbf{v}_1 = (A-\lambda_0 I)\mathbf{v}_2$. Then $\mathbf{v}_1$ is a nonzero eigenvector. Since $\mathbf{v}_2 \notin NS(A-\lambda_0 I)$, the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Then since $\mathbf{v}_1, \mathbf{v}_2 \in NS(A-\lambda_0 I)^2$ but $\mathbf{v}_3 \notin NS(A-\lambda_0 I)^2$, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and hence a basis for $\mathbb{C}^3$. We can now compute the matrix for A relative to this basis:

$$\begin{aligned} A\mathbf{v}_1 &= \lambda_0\mathbf{v}_1 + 0\cdot\mathbf{v}_2 + 0\cdot\mathbf{v}_3 \\ A\mathbf{v}_2 &= \mathbf{v}_1 + \lambda_0\mathbf{v}_2 + 0\cdot\mathbf{v}_3 \\ A\mathbf{v}_3 &= 0\cdot\mathbf{v}_1 + \mathbf{v}_2 + \lambda_0\mathbf{v}_3. \end{aligned}$$

Therefore the matrix relative to this basis is

$$J = \begin{pmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{pmatrix}.$$

As in the 2×2 case, this entire matrix is a single Jordan block. Then if $S = (\mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3)$, $A = SJS^{-1}$.

Example. Let

$$A = \frac{1}{3}\begin{pmatrix} 5 & -1 & -1 \\ 0 & 0 & 3 \\ 1 & -2 & 4 \end{pmatrix}.$$

Then the characteristic polynomial is $\det(\lambda I - A) = (\lambda - 1)^3$. The eigenspace for $\lambda = 1$ is

$$NS(A-I) = NS(3A-3I) = NS\begin{pmatrix} 2 & -1 & -1 \\ 0 & -3 & 3 \\ 1 & -2 & 1 \end{pmatrix} = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

For this kind of matrix, a good approach is to guess something for $\mathbf{v}_3$ and then compute $\mathbf{v}_2$ and $\mathbf{v}_1$. As long as $\mathbf{v}_1 \neq \mathbf{0}$, the original guess for $\mathbf{v}_3$ was valid. Let $\mathbf{v}_3 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$. Then $\mathbf{v}_2 = (A-I)\mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{v}_1 = (A-I)\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. If

$$S = (\mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

and

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

then $A = SJS^{-1}$.