MATH 54 Lecture Notes 18

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1 Linear Recurrence Relations

Recall the definition of the Fibonacci sequence:

$$f_{n+2} = f_{n+1} + f_n$$

for all $n \geq 0$, $f_0 = 0$, $f_1 = 1$. This gives the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

We can think of the Fibonacci sequence in terms of vectors:

$$\mathbf{f}_n = \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$$

for all $n \geq 1$. Then we have a sequence of vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ 5 \end{pmatrix}, \ldots$$

If we can write a formula for $\mathbf{f}_n$ in terms of n, then we'll have a formula for f_n in terms of n, which is simply the first entry of $\mathbf{f}_n$. Note that for all $n \geq 1$,

$$\mathbf{f}_{n+1} = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} f_n + f_{n-1} \\ f_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{f}_n.$$

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\mathbf{f}_{n+1} = A\mathbf{f}_n = A^2\mathbf{f}_{n-1} = A^3\mathbf{f}_{n-2} = \cdots = A^n\mathbf{f}_1.$$

In order to write down a formula for A^n, first we need to diagonalize A. Here is the characteristic polynomial of A:

$$\chi_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - \lambda - 1.$$

The roots are $\frac{1\pm\sqrt{5}}{2}$. Let $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. Then the eigenspace for λ_1 is

$$NS(A - \lambda_1 I) = NS \begin{pmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{pmatrix} = \text{Span}\left\{ \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \right\}$$

and the eigenspace for λ_2 is

$$NS(A - \lambda_2 I) = NS \begin{pmatrix} 1-\lambda_2 & 1 \\ 1 & -\lambda_2 \end{pmatrix} = \text{Span}\left\{ \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix} \right\}.$$

Then if $S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then $A = S\Lambda S^{-1}$, so $A^n = S\Lambda^n S^{-1}$. Then

$$\begin{aligned}
\mathbf{f}_{n+1} &= A^n \mathbf{f}_1 \\
&= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n \\ -\lambda_2^n \end{pmatrix} \\
&= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{pmatrix}.
\end{aligned}$$

Therefore

$$f_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}.$$

Now, using the fact that $\left|\frac{\lambda_2}{\lambda_1}\right| < 1$, we can compute

$$\begin{aligned}
\lim_{n\to\infty} \frac{f_{n+1}}{f_n} &= \lim_{n\to\infty} \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} \\
&= \lim_{n\to\infty} \frac{\lambda_1 - \lambda_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n}{1 - \left(\frac{\lambda_2}{\lambda_1}\right)^n} \\
&= \lambda_1.
\end{aligned}$$

We can use this same approach to compute closed form formulas for the terms of any sequence defined by a linear recurrence relation:

$$g_{n+r} = a_{r-1}g_{n+r-1} + a_{r-2}g_{n+r-2} + \cdots + a_1 g_{n+1} + a_0 g_n$$

where $a_0, a_1, \ldots, a_{r-1}$ are fixed scalars. The only thing that will change compared to the above computations for the Fibonacci seuqnce is which particular matrix is used to write the recurrence relation as a matrix equation.