MATH 54 Lecture Notes 17

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1 Spectral Theorem

Let A be a symmetric matrix with real entries, and let λ be an eigenvalue of A with nonzero eigenvector $\mathbf{v}$. Then

$$\begin{aligned}\lambda\langle \mathbf{v}, \mathbf{v}\rangle &= \langle \lambda\mathbf{v}, \mathbf{v}\rangle \\ &= \langle A\mathbf{v}, \mathbf{v}\rangle \\ &= \langle \mathbf{v}, A^T\mathbf{v}\rangle \\ &= \langle \mathbf{v}, A\mathbf{v}\rangle \\ &= \langle \mathbf{v}, \lambda\mathbf{v}\rangle \\ &= \overline{\lambda}\langle \mathbf{v}, \mathbf{v}\rangle.\end{aligned}$$

Therefore $(\lambda - \overline{\lambda})\langle \mathbf{v}, \mathbf{v}\rangle = 0$. Since $\mathbf{v} \neq \mathbf{0}$, $\langle \mathbf{v}, \mathbf{v}\rangle \neq 0$. Therefore $\lambda - \overline{\lambda} = 0$. That is, λ is real. It is also a theorem that A is diagonalizable (though we won't prove this part). All of these facts together are often referred to as the spectral theorem.

Now let A again be real and symmetric, with $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$, where $\lambda_1 \neq \lambda_2$. Then λ_1 and λ_2 are both real, and

$$\begin{aligned}(\lambda_1 - \lambda_2)\langle \mathbf{v}_1, \mathbf{v}_2\rangle &= \lambda_1\langle \mathbf{v}_1, \mathbf{v}_2\rangle - \lambda_2\langle \mathbf{v}_1, \mathbf{v}_2\rangle \\ &= \langle \lambda_1\mathbf{v}_1, \mathbf{v}_2\rangle - \langle \mathbf{v}_1, \lambda_2\mathbf{v}_2\rangle \\ &= \langle A\mathbf{v}_1, \mathbf{v}_2\rangle - \langle \mathbf{v}_1, A\mathbf{v}_2\rangle \\ &= \langle A\mathbf{v}_1, \mathbf{v}_2\rangle - \langle A\mathbf{v}_1, \mathbf{v}_2\rangle.\end{aligned}$$

This shows that, for any real symmetric matrix, eigenvectors from distinct eigenspaces are always orthogonal. Due to this fact, we can come up with an orthogonal matrix that diagonalizes A whenever A is real and symmetric.

Exercise 1.

Exercise 7.