MATH 54 Lecture Notes 16

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1 Traces, Determinants, and Eigenvalues

The *trace* of an $n \times n$ matrix A is the sum of its diagonal entries.

The sum of the eigenvalues of any matrix is equal to its trace, and the product of the eigenvalues of any matrix is equal to its determinant. These sums and products must be computed with multiplicity, as illustrated by the above matrix B.

In particular, a square matrix is invertible if and only if it does not have zero as one of its eigenvalues.

2 Eigenvalues and Polynomials

Let $\mathbf{v}$ be a nonzero eigenvector for A with eigenvalue λ. Then

$$A^2\mathbf{v} = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda^2\mathbf{v}$$

and

$$A^3\mathbf{v} = A(A^2\mathbf{v}) = A(\lambda^2\mathbf{v}) = \lambda^2(A\mathbf{v}) = \lambda^3\mathbf{v}.$$

This pattern continues, and in general, we get that $A^k\mathbf{v} = \lambda^k\mathbf{v}$ for all $k \geq 0$.

Now suppose A is invertible. Then $\lambda \neq 0$, and

$$\mathbf{v} = A^{-1}A\mathbf{v} = \lambda A^{-1}\mathbf{v}$$

so that

$$A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}.$$

Therefore our formula $A^k\mathbf{v} = \lambda^k\mathbf{v}$ holds for all k as long as A is invertible (otherwise the left-hand side is not defined for negative values of k).

Now let $a \in \mathbb{C}$. Then

$$(A - aI)\mathbf{v} = A\mathbf{v} - a\mathbf{v} = \lambda\mathbf{v} - a\mathbf{v} = (\lambda - a)\mathbf{v}.$$

In general, let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Then we define $f(A)$ as

$$f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n.$$

Then

$$\begin{aligned}
f(A)\mathbf{v} &= (a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n)\mathbf{v} \\
&= a_0\mathbf{v} + a_1 A\mathbf{v} + a_2 A^2\mathbf{v} + \cdots + a_n A^n\mathbf{v} \\
&= a_0\mathbf{v} + a_1 \lambda\mathbf{v} + a_2 \lambda^2\mathbf{v} + \cdots + a_n \lambda^n\mathbf{v} \\
&= (a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n)\mathbf{v} \\
&= f(\lambda)\mathbf{v}.
\end{aligned}$$

So, any polynomial of a matrix A has the same eigenvectors, and the new eigenvalues are obtained by plugging in the old eigenvalues into the same polynomial.

Example. Suppose $A^2+A-5I=0$. Then we'll prove that A is invertible. Let $A\mathbf{v}=\lambda\mathbf{v}$ (that is, λ is some eigenvalue of A, and $\mathbf{v}$ is some nonzero eigenvector corresponding to that eigenvalue). Then

$$(\lambda^2+\lambda-5)\mathbf{v} = (A^2+A-5I)\mathbf{v} = 0\cdot\mathbf{v} = \mathbf{0}.$$

Since $\mathbf{v}\neq\mathbf{0}$, $\lambda^2+\lambda-5=0$. Therefore $\lambda\neq 0$, so A is invertible.

Now let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be nonzero eigenvectors for a square matrix A with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. We will show that these eigenvectors are linearly independent. Suppose

$$\sum_{i=1}^{n} a_i\mathbf{v}_i = \mathbf{0}$$

is a nontrivial linear dependence relation among the eigenvectors. Without loss of generality, we may suppose that $a_1\neq 0$. Let

$$g(x) = \prod_{i=2}^{n}(x-\lambda_i).$$

Then

$$\begin{aligned}
\mathbf{0} &= g(A)\mathbf{0} \\
&= \sum_{i=1}^{n} a_i g(A)\mathbf{v}_i \\
&= \sum_{i=1}^{n} a_i g(\lambda_i)\mathbf{v}_i \\
&= \sum_{i=1}^{n} a_i \left(\prod_{j=2}^{n}(\lambda_j-\lambda_i)\right)\mathbf{v}_i \\
&= a_1\left(\prod_{j=2}^{n}(\lambda_j-\lambda_1)\right)\mathbf{v}_1 \\
&\neq \mathbf{0}.
\end{aligned}$$

This is a contradiction, so the eigenvectors are linearly independent. This result tells us that if we make a list consisting of bases of all the different eigenspaces of a matrix, then that list is linearly independent.

For example, suppose B is a matrix with an eigenvector $\mathbf{v}_1$ with eigenvalue λ_1, and linearly independent eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$ for the eigenvalue λ_2. Then suppose

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}.$$

Since $\mathbf{v}_1 \in NS(A-\lambda_1 I)$ and $a_2\mathbf{v}_2 + a_3\mathbf{v}_3 \in NS(A-\lambda_2 I)$, the previous theorem tells us that $a_1 = 0$ and $a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$. Then since $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, $a_2 = a_3 = 0$.

3 Diagonalization

Suppose A is a 3×3 matrix with linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ corresponding to eigenvalues λ_1, λ_2, and λ_3 (not necessarily distinct). Then these three eigenvectors form a basis for $\mathbb{R}^3$, and we can write A relative to this basis. Since

$$\begin{aligned} A\mathbf{v}_1 &= \lambda_1\mathbf{v}_1 + 0 \cdot \mathbf{v}_2 + 0 \cdot \mathbf{v}_3 \\ A\mathbf{v}_2 &= 0 \cdot \mathbf{v}_1 + \lambda_2\mathbf{v}_2 + 0 \cdot \mathbf{v}_3 \\ A\mathbf{v}_3 &= 0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 + \lambda_3\mathbf{v}_3, \end{aligned}$$

A relative to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a diagonal matrix with the eigenvalues of A as the diagonal entries.

To diagonalize a matrix means to give a similarity transformation between it and a diagonal matrix. Let A be an $n \times n$ matrix, and suppose $A = P^{-1}BP$. If the columns of P are eigenvectors of A, then B will be diagonal. A matrix is called diagonalizable if it has a basis of eigenvectors.

Exercise 19.

Exercise 24.

Exercise 26.