MATH 54 Lecture Notes 15

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1 Determinants

The determinant of a 1×1 matrix is simply the unique entry of that matrix. For an $n \times n$ matrix, the cofactor C_{ij} of the matrix is given by

$$C_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the determinant of the matrix with the i-th row and j-th column removed. Then

$$\det A = \sum_{i=1}^{n} (A)_{ij} C_{ij}$$

for any j.

Computing 1×1 determinants is easy. See that this definition agrees with our previous notion of determinants for 2×2 and 3×3 matrices.

We only have the concept of a determinant for square matrices.

Some facts about determinants:

- $(\det A)(\det B) = \det(AB)$
- $\det(A^T) = \det A$
- The determinant of any lower triangular matrix or upper triangular matrix is the product of the elements on the diagonal.
- The determinant of any invertible matrix is nonzero. We can see this from the fact that the determinant of any elementary matrix is nonzero.
- Any matrix with nonzero determinant is invertible. This can be seen from the explicit formula for A^{-1}:

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A}.$$

Exercise 11.

Exercise 25. Sometimes the determinant is a polynomial.

Exercise 26. In general, if each entry of a matrix is a linear polynomial, the determinant of an $n \times n$ matrix with be a polynomial of degree n.

2 Eigenvalues and Eigenvectors

2.1 Definitions

Let A be an $n \times n$ matrix. Then an *eigenvalue* for A is a scalar $\lambda \in \mathbb{C}$ such that there exists a nonzero vector $\mathbf{v} \in \mathbb{C}^n$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Any such vector $\mathbf{v}$ is called an *eigenvector*. We require $\mathbf{v}$ to be nonzero because otherwise every scalar is an eigenvalue. The above equation is equivalent to

$$(A - \lambda I)\mathbf{v} = A\mathbf{v} - \lambda\mathbf{v} = 0.$$

So we are looking for values of λ such that $NS(A - \lambda I)$ is nontrivial. This is equivalent to saying that the matrix $A - \lambda I$ is singular, which is in turn equivalent to

$$\det(\lambda I - A) = 0.$$

The expression $\det(\lambda I - A)$ is a polynomial of degree n in the variable λ. It is called the characteristic polynomial of A, which is written $\chi_A(\lambda)$. Thus, for any matrix A, the eigenvalues of A are the roots of the polynomial $\chi_A(\lambda)$.

Since the set of eigenvectors for a given eigenvector λ is equal to $NS(\lambda I - A)$, it is a subspace of $\mathbb{C}^n$. This subspace is called the eigenspace associated to λ.

2.2 Examples

It's easy to read off the eigenvalues of a diagonal matrix. Take, for example,

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then $\chi_A(\lambda) = (\lambda - 2)(\lambda + 1)(\lambda + 3)$. The eigenspaces are each one-dimensional.

It's also easy to compute the eigenvalues of upper-triangular matrices. Take the matrix

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\chi_B(\lambda) = (\lambda - 2)^2(\lambda - 1)$. In this case, there are two eigenspaces, each of dimension 1. Any $n \times n$ matrix has at most n linearly independent eigenvectors, but here's an example of a 3×3 matrix with only two linearly independent eigenvectors.

Exercise 13.

Exercise 26. Here we show that if $A\mathbf{v} = \lambda\mathbf{v}$ and A is invertible, then $A^k\mathbf{v} = \lambda^k\mathbf{v}$ for any integer k. The formula still holds if A is not invertible as long as $k \geq 0$.

Exercise 28. Here we show that if $A\mathbf{v} = \lambda\mathbf{v}$, then $(A - aI)\mathbf{v} = (\lambda - a)\mathbf{v}$.

Exercise 35.

3 Traces and Determinants

The *trace* of an $n \times n$ matrix A is the sum of its diagonal entries.

The sum of the eigenvalues of any matrix is equal to its trace, and the product of the eigenvalues of any matrix is equal to its determinant. These sums and products must be computed with multiplicity, as illustrated by the above matrix B.