MATH 54 Lecture Notes 14

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1 Orthogonal Matrices

A collection of vectors is called *orthonormal* if the vectors are orthogonal and each of them has length 1. We can get an orthonormal basis from an orthogonal basis by dividing each vector by its length. For instance, take the orthogonal basis $(2,1),(-1,2)$ of $\mathbb{R}^2$.

A matrix is called *orthogonal* if it is square and its columns are orthonormal. Note the discrepancy of terminology! The discrepancy exists because the idea of a matrix with orthogonal columns is less useful.

Let $Q = (\mathbf{q}_1 \mid \mathbf{q}_2 \mid \cdots \mid \mathbf{q}_n)$ be an $n \times n$ orthogonal matrix with real entries. Then the (i,j)-th entry of Q^TQ is

$$\sum_{k=1}^{n} q_{ki}q_{kj} = \sum_{k=1}^{n} q_{ki}\overline{q_{kj}} = \langle \mathbf{q}_i, \mathbf{q}_j \rangle$$

so that $Q^TQ = I$. In fact, for a square matrix Q with real entries, Q being orthogonal is equivalent to $Q^T = Q^{-1}$.

Example. $A = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

Now let Q be real and orthogonal, with dimensions $n \times n$. Then for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\langle Q\mathbf{u}, Q\mathbf{v} \rangle = \langle Q^TQ\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

and

$$\|Q\mathbf{u}\| = \sqrt{\langle Q\mathbf{u}, Q\mathbf{u} \rangle} = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \|\mathbf{u}\|.$$

That is, Q preserves lengths and angles.

Example. Let $\mathbf{u} = (1,0,0)^T$ and $\mathbf{v} = (0,1,1)^T$. What happens when we multiply by A?

2 The matrix for a linear transformation relative to choices of bases

2.1 General Case

We have already seen that a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ corresponds to an $m \times n$ matrix. Let V be n-dimensional and W be m-dimensional. Then if we choose bases for V and W, we can produce a matrix for any linear transformation $V \to W$ relative to those bases.

Let $T : V \to W$ be a linear transformation, let $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for V, and let $C = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m\}$ be a basis for W. Then the matrix for T relative to B and C, which we write $[T]_{BC}$, has the property that

$$[T]_{BC}[\mathbf{x}]_B = [T(\mathbf{x})]_C$$

for all $\mathbf{x} \in V$. Therefore

$$[T]_{BC}\mathbf{e}_i = [T]_{BC}[\mathbf{b}_i]_B = [T(\mathbf{b}_i)]_C$$

so that

$$[T]_{BC} = ([T(\mathbf{b}_1)]_C \mid [T(\mathbf{b}_2)]_C \mid \cdots \mid [T(\mathbf{b}_n)]_C)\,.$$

Example. Let $D : P_3 \to P_2$ be the operation of differentation, and let $B = \{1, x-1, (x-1)^2, (x-1)^3\}$ and $C = \{1, x-1, (x-1)^2\}$. Then since

$$\begin{aligned} D(1) &= 0 \\ D(x-1) &= 1 \\ D((x-1)^2) &= 2(x-1) \\ D((x-1)^3) &= 3(x-1)^2 \end{aligned}$$

the matrix for D relative to these two bases is

$$[D]_{BC} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

2.2 Square matrices and similarity

When T is a linear transformation from V to V and B is a basis for V, we write $[T]_B$ as an abbreviation for $[T]_{BB}$.

Let B and C be two bases for V, and let P be the transition matrix from C to B. Then P^{-1} is the transition matrix from B to C, and for any $\mathbf{x} \in V$,

$$[T]_C[\mathbf{x}]_C = [T(\mathbf{x})]_C$$

and

$$(P^{-1}[T]_B P)[\mathbf{x}]_C = P^{-1}[T]_B[\mathbf{x}]_B = P^{-1}[T(\mathbf{x})]_B = [T(\mathbf{x})]_C.$$

Therefore

$$[T]_C = P^{-1}[T]P.$$

Example. Let

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then we can compute f_A relative to the basis $C = \{(1,-1)^T, (1,0)^T\}$. If B is the standard basis in $\mathbb{R}^2$, then $[f_A]_B = A$. Then if P is the transition matrix from C to B,

$$[f_A]_C = P^{-1}AP.$$

Since B is the standard basis, P is simply the matrix whose columns are the elements of C, so

$$\begin{aligned} [f_A]_C &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Let A and B be $n \times n$ matrices such that there exists an invertible $n \times n$ matrix P such that

$$A = P^{-1}BP.$$

Then we say that A and B are *similar*.

The matrices for the same linear transformation relative to two different bases (i.e. $[T]_B$ and $[T]_C$) are always similar. Two similar matrices share many properties, such as their rank being the same, since rank is preserved by multiplication by an invertible matrix on either the right or the left.

Exercise 23.

Exercise 24.

Exercise 29.

Exercise 31. Since $A = I^{-1}AI$, A is similar to itself.

Exercise 32. If $A = P^{-1}DP$, then $D = PAP^{-1} = (P^{-1})^{-1}AP^{-1}$.

Exercise 33. If $A = P^{-1}DP$ and $D = Q^{-1}EQ$, then

$$A = P^{-1}(Q^{-1}EQ)P = (QP)^{-1}E(QP).$$

Therefore A is similar to E.

These three exercises show that similarity is an equivalence relation. This means that the relation of similarity divides the set of all matrices into disjoint classes of matrices, where all the matrices in each class are similar to each other.

Exercise 34. The identity matrix is only similar to itself.

Exercise 35. If $A = P^{-1}BP$,

$$A^2 = P^{-1}BPP^{-1}BP = P^{-1}B^2P.$$

The same argument shows that A^k is similar to B^k for all positive integers k.