MATH 54 Lecture Notes 13

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July 18, 2007

1 Gram-Schmidt and Projection Example

Consider the subspace

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$

of $\mathbb{R}^3$. Suppose we want to project vectors in $\mathbb{R}^3$ to V. The first step is to write down some basis for V, such as

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Next we need to apply Gram-Schmidt to this basis in order to obtain an orthogonal basis for V.

$$\mathbf{p}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{p}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{2} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1 \\ 1/2 \end{pmatrix}$$

Now that we have an orthogonal basis $\{\mathbf{p}_1, \mathbf{p}_2\}$ for V, we can project an arbitrary vector $\mathbf{w} \in \mathbb{R}^3$ to V by the formula

$$\text{proj}_V \mathbf{w} = \text{proj}_{\mathbf{p}_1}\mathbf{w} + \text{proj}_{\mathbf{p}_2}\mathbf{w}.$$

Since projection to V is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$, we can write down the corresponding 3×3 matrix by applying the transformation to the three elements of the standard basis for $\mathbb{R}^3$.

$$\begin{aligned}
\text{proj}_V \mathbf{e}_1 &= \frac{1}{2}\mathbf{p}_1 + \frac{1}{2}\cdot\frac{2}{3}\mathbf{p}_2 = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 1/6 \\ -1/3 \\ 1/6 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix} \\
\text{proj}_V \mathbf{e}_2 &= 0\cdot\mathbf{p}_1 - \frac{2}{3}\mathbf{p}_2 = \begin{pmatrix} -1/3 \\ 2/3 \\ -1/3 \end{pmatrix} \\
\text{proj}_V \mathbf{e}_3 &= -\frac{1}{2}\mathbf{p}_1 + \frac{1}{2}\cdot\frac{2}{3}\mathbf{p}_2 = \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 1/6 \\ -1/3 \\ 1/6 \end{pmatrix} = \begin{pmatrix} -1/3 \\ -1/3 \\ 2/3 \end{pmatrix}
\end{aligned}$$

Therefore the matrix for orthogonal projection to V is

$$\frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

For example, the orthogonal projection of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ to V is

$$\frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix}.$$

2 Integral-Square Approximations

Recall the Taylor expansion of a function $f(x)$ near $x = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n.$$

For certain kinds of functions, the infinite series on the right is actually equal to the original function on some open interval around $x = 0$. If we cut off the series at some point, then we get an approximation for $f(x)$ which is very good close to $x = 0$ but quickly gets worse as we move away from there.

Consider the function $f(x) = \cos x$ on the interval $[-\pi/2, \pi/2]$. The approximation in P_2 for $f(x)$ given to us by the Taylor series is $g(x) = 1 - \frac{x^2}{2}$. This approximation is good near the origin, but not as good across the entire interval.

In order to get an approximation which is better across a larger interval, we will use the inner product on $C[a,b]$ given by

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\, dx.$$

If we approximate a function $f \in C[a,b]$ by another function f_0, then the square of the magnitude of the error in terms of this inner product is

$$\|f - f_0\|^2 = \langle f - f_0, f - f_0 \rangle = \int_a^b |f(x) - f_0(x)|^2 \, dx.$$

Hence we will be looking for an approximation which minimizes the integral of the square of the error.

In general, the n-th order integral-square approximation of a function $f \in C[a,b]$ is given by $\mathrm{proj}_{P_n} f$ with respect to the above inner product.

Now let's find the second-order integral-square approximation for $f(x) = \cos x$ on the interval $[-\pi/2, \pi/2]$. First we need an orthogonal basis for P_2 with respect to this inner product, which we can obtain by applying Gram-Schmidt to the basis $\{1, x, x^2\}$.

$$p_1(x) = 1$$

$$p_2(x) = x - \mathrm{proj}_1 x = x - \frac{\int_{-\pi/2}^{\pi/2} x \, dx}{\int_{-\pi/2}^{\pi/2} dx} = x$$

$$p_3(x) = x^2 - \mathrm{proj}_1 x^2 - \mathrm{proj}_x x^2 = x^2 - \frac{\int_{-\pi/2}^{\pi/2} x^2 \, dx}{\int_{-\pi/2}^{\pi/2} dx} - \frac{\int_{-\pi/2}^{\pi/2} x^3 \, dx}{\int_{-\pi/2}^{\pi/2} x^2 \, dx} \cdot x = x^2 - \frac{\pi^2}{12}$$

Now we can compute the projection of $\cos x$:

$$\begin{aligned} \mathrm{proj}_{P_2} \cos x &= \mathrm{proj}_1 \cos x + \mathrm{proj}_x \cos x + \mathrm{proj}_{x^2 - \frac{\pi^2}{12}} \cos x \\ &= \frac{\int_{-\pi/2}^{\pi/2} \cos x \, dx}{\pi} + \frac{\int_{-\pi/2}^{\pi/2} x \cos x \, dx}{\int_{-\pi/2}^{\pi/2} x^2 \, dx} \cdot x + \frac{\int_{-\pi/2}^{\pi/2} \left(x^2 - \frac{\pi^2}{12}\right) \cos x \, dx}{\int_{-\pi/2}^{\pi/2} \left(x^2 - \frac{\pi^2}{12}\right)^2 \, dx} \cdot \left(x^2 - \frac{\pi^2}{12}\right) \\ &= \frac{2}{\pi} + \frac{60\pi^2 - 720}{\pi^5} \left(x^2 - \frac{\pi^2}{12}\right). \end{aligned}$$

This approximation is not as good as the one obtained from the Taylor series near $x = 0$, but it fares better across the whole interval $[-\pi/2, \pi/2]$. On this interval, the approximation is actually equal to $\cos x$ at four different place, whereas the Taylor approximation is only equal to $\cos x$ at $x = 0$.