MATH 54 Lecture Notes 11

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July 16, 2007

1 Inner Product Spaces

1.1 Definition

Let V be a $\mathbb{C}$-vector space. Then an *inner product* on V is any function from ordered pairs in V to $\mathbb{C}$, $\langle\ ,\ \rangle$, with the following properties.

- Linear in the first variable. That is, for any $\alpha \in \mathbb{C}$ and any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,
$$\langle \alpha\mathbf{u}, \mathbf{w}\rangle = \alpha\langle \mathbf{u}, \mathbf{w}\rangle$$
and
$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle.$$
- For any $\mathbf{u}, \mathbf{v} \in V$,
$$\langle \mathbf{u}, \mathbf{v}\rangle = \overline{\langle \mathbf{v}, \mathbf{u}\rangle}.$$
- For any $\mathbf{u} \in V$, $\langle \mathbf{u}, \mathbf{u}\rangle \geq 0$ (in particular, the inner product of any vector with itself is a real number), and $\langle \mathbf{u}, \mathbf{u}\rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

As a consequence of these three rules, any inner product will be *conjugate linear* in the second variable. That is,
$$\langle \mathbf{u}, \alpha\mathbf{w}\rangle = \overline{\langle \alpha\mathbf{w}, \mathbf{u}\rangle} = \overline{\alpha\langle \mathbf{w}, \mathbf{u}\rangle} = \overline{\alpha}\langle \mathbf{u}, \mathbf{w}\rangle$$
and
$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle.$$

We can also have an inner product on an $\mathbb{R}$-vector space. In that case, the inner product of any two vectors should be a real number, and all above references to scalars apply only to real numbers. In this case, the above definition matches the one in the book.

A vector space with an inner product is called an *inner product space*. In any inner product space, we can define a notion of the length of a vector, or it's norm:
$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle}.$$
The last axiom in the definition of an inner product space guarantees that all vectors have non-negative, real length, and that only the zero vector has length zero.

1.2 Dot Product

The simplest example of an inner product is the dot product on $\mathbb{C}^n$. Given $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, the dot product of $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{n} a_i \overline{b_i}.$$

The length of $\mathbf{e}_i$ is 1.

Exercise 3.2.15. $\|(1, -2, 0, 2)\| = \sqrt{1+4+4} = 3.$

1.3 Convolution Integral

Another important example of an inner product is the convolution integral inner product on $C[a, b]$. This is defined as

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

for all $f, g \in C[a, b]$. It is clear that this satisfies the first two axioms. For the third, let $f \in C[a, b]$ be any nonzero continuous function. Then there exists $x_0 \in (a, b)$ such that $f(x_0) \neq 0$. Let $\epsilon = \frac{|f(x_0)|}{2}$. Then there exist $\delta > 0$ such that

- $a < x_0 - \delta < x_0 + \delta < b$; and
- $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

Then for all $x \in (x_0 - \delta, x_0 + \delta)$,

$$|f(x)| > |f(x_0)| - \epsilon = \frac{|f(x_0)|}{2}$$

so that

$$\begin{aligned}
\langle f, f \rangle &= \int_a^b f(x)\overline{f(x)}\, dx \\
&= \int_a^b |f(x)|^2\, dx \\
&\geq \int_{x_0-\delta}^{x_0+\delta} |f(x)|^2\, dx \\
&\geq \int_{x_0-\delta}^{x_0+\delta} \frac{|f(x_0)|^2}{4}\, dx \\
&= \frac{\delta |f(x_0)|^2}{2} \\
&> 0.
\end{aligned}$$

Then since it is clear that $\langle 0, 0 \rangle = 0$, the third axiom is satisfied.

2 One-Dimensional Projections

In any inner product space, we can project any vector $\mathbf{u}$ onto a nonzero vector $\mathbf{v}$ with the following formula:

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \cdot \mathbf{v}.$$

The result is a vector inside $\mathrm{Span}\{\mathbf{v}\}$.