MATH 54 Lecture Notes 10

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1 Transition Matrices

Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $C = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be two ordered bases for the same vector space V of dimension n. Then there is an invertible $n \times n$ matrix P such that

$$[\mathbf{x}]_C = P[\mathbf{x}]_B$$

for all $\mathbf{x} \in V$. To compute these matrices, we will use the following facts:

- The matrix $S = (\mathbf{b}_1 \mid \cdots \mid \mathbf{b}_n)$ is the transition matrix from B to the standard basis.
- Let $T = (\mathbf{c}_1 \mid \cdots \mid \mathbf{c}_n)$. Then T^{-1} is the transition matrix from the standard basis to C. Therefore $T^{-1}S$ is the transition matrix from B to C.

We can compute $T^{-1}S$ by starting with $(T \mid S)$ and then row reducing until the left side is I. Then $T^{-1}S$ will be on the right hand side.

Exercise 3.8.22. Here we are given bases $B = \{(1,1,1), (-2,-1,0), (2,1,2)\}$ and $C = \{(-6,-2,1), (-1,1,5), (-1,-1,1)\}$. Once we find the transition matrix, we can use it to compute $[\mathbf{x}]_C$ from $[\mathbf{x}]_B = (-3,2,4)$.

An alternative way of computing P is to compute the i-th column, which is

$$P\mathbf{e}_i = P[\mathbf{b}_i]_B = [\mathbf{b}_i]_C.$$

Exercise 3.8.26.

2 Linear Transformations

Let V and W be vector spaces. Then a function $T : V \to W$ is called a *linear transformation* if, for all $\mathbf{u}, \mathbf{v} \in V$ and all scalars α,

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$; and
- $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$.

The most basic example is matrix multiplication. Let $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, and A be some $m \times n$ matrix. Then there is a linear transformation $f_A : V \to W$ given by $f_A(\mathbf{u}) = A\mathbf{u}$ for all $\mathbf{u} \in V$.

Note that it is enough to check that $T(\alpha\mathbf{u}+\mathbf{v}) = \alpha T(\mathbf{u})+T(\mathbf{v})$ for all scalars α and all vectors $\mathbf{u}$ and $\mathbf{v}$ in the domain of T. This test shows that the first rule is satisfied by setting $\alpha = 1$. Then since the first rule implies that $T(\mathbf{0}) = \mathbf{0}$, setting $\mathbf{v} = \mathbf{0}$ shows that this test implies the second rule.

Any linear transformation must send $\mathbf{0}$ to $\mathbf{0}$. This gives a quick way to check whether something is a linear transformation, such as in exercise 34.

The range of a linear transformation $T : V \to W$ is a subspace of W.

Any linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by some $m \times n$ matrix. (We can replace $\mathbb{R}^n$ and $\mathbb{R}^m$ with $\mathbb{C}^n$ and $\mathbb{C}^m$, and it's still true.) To find the i-th column of this matrix, just check where the transformation sends $\mathbf{e}_i$, such as in exercise 13.

Exercise 18. Linear.

Exercise 20. Not linear. Take $(x, y) = (0, 1)$ and check scalar multiplication.

Exercise 23. Linear. Any function which sends every vector in a vector space to $\mathbf{0}$ is always linear.

Exercise 36. Linear.

Exercise 37. Linear.

Exercise 39. Not linear. Consider $T(2I)$.

Exercise 40. Not linear. Consider $T(2I)$.

Exercise 41. For part (a), J is linear. For part (b), we are told the domain is $C[a, b]$. As long as $b > a$, the range is then all of $\mathbb{R}$.