MATH 54 Lecture Notes 1

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1 Complex Numbers

1.1 Definition

A *complex number* has the form $a + bi$, where $a, b \in \mathbb{R}$ and $i^2 = -1$. The set of all complex numbers is denoted $\mathbb{C}$. Some examples of complex numbers are:

- i
- 5 (in fact, every real number is also a complex number)
- $2 + 3i$
- $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

Every complex number has what is called a *real part* and an *imaginary part.* In the complex number $a + bi$ above, the real part is a and the imaginary part is b (not bi). This is written $\mathrm{Re}(a + bi) = a$ and $\mathrm{Im}(a + bi) = b$. Here are the real and imaginary parts of all of the above complex numbers:

- $\mathrm{Re}(i) = 0$; $\mathrm{Im}(i) = 1$
- $\mathrm{Re}(5) = 5$; $\mathrm{Im}(5) = 0$
- $\mathrm{Re}(2 + 3i) = 2$; $\mathrm{Im}(2 + 3i) = 3$
- $\mathrm{Re}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{\sqrt{2}}{2}$; $\mathrm{Im}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{\sqrt{2}}{2}$

What it means for two complex numbers to be equal is that their real parts and their imaginary parts are equal.

1.2 Complex Conjugation

Let $z \in \mathbb{C}$. If $z = a + bi$, where $a, b \in \mathbb{R}$, then the *complex conjugate* of z, written $\overline{z}$, is equal to $a - bi$. Here are some examples of complex conjugation:

- $\overline{i} = -i$. For any *purely imaginary* complex number (that is, a complex number where the real part is 0), complex conjugation is the same as negation.

- $\overline{5} = 5$. Any real number is equal to its own complex conjugate.
- $\overline{(2+3i)} = 2 - 3i$.
- $\overline{\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

For any complex number z, $\mathrm{Re}(\overline{z}) = \mathrm{Re}(z)$ and $\mathrm{Im}(\overline{z}) = -\mathrm{Im}(z)$.

1.3 Complex Arithmetic

Just like with real numbers, we can add, subtract, multiply, and divide complex numbers, except we can't divide by zero. To do so, first treat the complex numbers as polynomials in the variable i. Then use the fact that $i^2 = -1$ to eliminate higher powers of i. Some examples:

- $(2+3i) + (5-2i) = 2 + 3i + 5 - 2i = 7 + i$.
- $(3+i) - (4-2i) = 3 + i - 4 + 2i = 7 + 3i$.
- $(5) \cdot (1+i) = 5 + 5i$.
- $(1-2i) \cdot (3+7i) = 3 + 7i - 6i - 14i^2 = 3 + i - 14i^2 = 3 + i + 14 = 17 + i$.
- $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = \frac{1}{2} + 2 \cdot \frac{1}{2}i + \frac{1}{2}i^2 = \frac{1}{2} + i - \frac{1}{2} = i$.
- $(3+i) \cdot (3-i) = 9 - i^2 = 10$.

The last example above illustrates a principle that will help us divide complex numbers: the product of any complex number and its complex conjugate is a nonnegative real number. Some examples of division:

- $\frac{3-4i}{i} = \frac{3-4i}{i} \cdot \frac{-i}{-i} = \frac{-3i+4i^2}{-i^2} = -4 - 3i$.
- $\frac{1}{2+i} = \frac{1}{2+i} \cdot \frac{2-i}{2-i} = \frac{2-i}{4-i^2} = \frac{2-i}{5} = \frac{2}{5} - \frac{1}{5}i$.
- $\frac{1-i}{1+i} = \frac{1-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-2i+i^2}{1-i^2} = \frac{-2i}{2} = -i$.

Now let $f(z) = \frac{z^2}{z+1} - z + 1$, where $z \in \mathbb{C}$. Then

$$f(i) = \frac{i^2}{i+1} - i + 1 = \frac{-1}{i+1} \cdot \frac{1-i}{1-i} - i + 1 = \frac{-1+i}{2} - i + 1 = \frac{1}{2} - \frac{1}{2}i.$$

2 Systems of Linear Equations

2.1 Linear Equations

A linear equation is one in which each side of the equation consists of different variables multiplied by constants and added together. Some examples of linear equations:

- $x = 5$
- $3x + 4y = 0$
- $3x + 8y = 9z - 1$

Some examples of equations which are *not* linear:

- $\sin x = 1$, because the variable x is being plugged into the function sin.
- $xy = 4$, because variables are being multiplied together.
- $x^2 = 1$, because the variable x is being multiplied by itself. In linear equations, variables can only be multiplied by constants.

Let x and y be variables, and let k be a constant. Is the equation $(\sin k)\cdot x + y = 1$ linear?

Constants are also allowed to be complex numbers. So, for example, $ix = y$ and $x + y + z = 5 + i$ are linear equations.

2.2 Solving Systems of Linear Equations

2.2.1 Systems with a Unique Solution

Suppose we have two linear equations,

$$\begin{aligned} 2x + 3y &= 7 \\ 3x - y &= 2 \end{aligned}$$

and we would like to know which values of x and y satisfy both equations simultaneously. There are three operations we can perform on a system of linear equations to try and simplify it:

- multiplying a single equation by some nonzero constant;
- rearranging the equations; and
- adding a constant multiple of some equation to another equation.

For the above system of equations, we can let our first step be to make one of the x coefficients 1,

$$\begin{aligned} x + \frac{3}{2}y &= \frac{7}{2} \\ 3x - y &= 2. \end{aligned}$$

Then we can subtract the appropriate multiple of the first equation from the second equation to make its x coefficient 0,

$$\begin{aligned} x + \frac{3}{2}y &= \frac{7}{2} \\ -\frac{11}{2}y &= -\frac{17}{2}. \end{aligned}$$

Then we know that $y = \frac{17}{11}$. We can then backsubstitute this value into the first equation to obtain

$$x = \frac{7}{2} - \frac{3}{2} \cdot \frac{17}{11} = \frac{13}{11}.$$

Suppose instead we had three equations in three unknowns,

$$\begin{aligned} x + 2y - z &= 1 \\ x + y + 2z &= 0 \\ x + 3y - 3z &= 4. \end{aligned}$$

Then we can subtract the top equation from the other two to make all x coefficients 0 except for the one in the top equation,

$$\begin{aligned} x + 2y - z &= 1 \\ -y + 3z &= -1 \\ y - 2z &= 3. \end{aligned}$$

Then we can add the middle equation to the bottom one to make the bottom y coefficient 0,

$$\begin{aligned} x + 2y - z &= 1 \\ -y + 3z &= -1 \\ z &= 2. \end{aligned}$$

Then we can use backsubstitution to calculuate $y = 1 + 3 \cdot 2 = 7$ and $x = 1 + 2 - 2 \cdot 7 = -11$.

This also works when the equations have complex coefficients. Consider, for example, the system

$$\begin{aligned} ix + y &= 4 \\ -x + y &= 3. \end{aligned}$$

First we should make the x coefficient on the first equation equal to 1. When the coefficient was a nonzero real number, we just divided the equation by that number. However, we can also divide by any nonzero complex number. In this case, $\frac{1}{i} = \frac{1}{i} \cdot \frac{-i}{-i} = -i$. So, we get

$$\begin{aligned} x - iy &= -4i \\ -x + y &= 3. \end{aligned}$$

Now we can add the top equation to the bottom one to obtain

$$\begin{aligned} x - iy &= -4i \\ (1-i)y &= 3 - 4i. \end{aligned}$$

Then $y = \frac{3-4i}{1-i} = \frac{3-4i}{1-i} \cdot \frac{1+i}{1+i} = \frac{7-i}{2}$. Now we can compute $x = i\left(\frac{7-i}{2}\right) - 4i = \frac{1}{2} - \frac{1}{2}i$.

2.2.2 Inconsistent Systems

Not every system of linear equations has a solution. Consider the system

$$\begin{aligned} x + y &= 0 \\ x + y &= 1. \end{aligned}$$

No matter what x and y are, $x + y$ cannot be both 0 and 1 at the same time. Therefore this system has no solution. Such a system is called *inconsistent.* If we apply the above technique to this system, we get

$$\begin{aligned} x + y &= 0 \\ 0 &= 1. \end{aligned}$$

Every inconsistent system of linear equations reduces to a system with the equation $0 = 1$ by the above three types of operations.

2.2.3 Systems with Infinitely Many Solutions

Consider the system

$$\begin{aligned} x + y - z &= 4 \\ 2x - y + 3z &= 1. \end{aligned}$$

This system cannot have a unique solution, since there are more equations than unknowns. (Sometimes there is still not a unique solution even if the number of equations is equal to the number of unknowns.) The system reduces to

$$\begin{aligned} x + y - z &= 4 \\ -3y + 5z &= -7. \end{aligned}$$

Here z is a *free variable.* That is, we can choose any value of z we like, and there is a solution to the whole system with that value for z. Then $y = \frac{7}{3} + \frac{5}{3}z$, so $x = 4 + z - y = 4 + z - \frac{7}{3} - \frac{5}{3}z = \frac{5}{3} - \frac{2}{3}z$.

2.2.4 Homogeneous Systems

A linear equation is called *homogeneous* if setting all the variables to 0 gives a solution. This is equivalent to saying that the constant term of the equation is 0. A system of linear equations is called homogeneous if it consists entirely of homogeneous linear equations. An example:

$$\begin{aligned} x + 2y &= 0 \\ -x + y &= 0. \end{aligned}$$

Any homogeneous linear system is consistent, since there is always a solution where all the variables are equal to 0. This solution is called the trivial solution.

2.3 Augmented Matrices

An equivalent way of looking at a system of linear equations is as an augmented matrix. Take the system with complex coefficients from earlier:

$$\begin{aligned} ix + y &= 4 \\ -x + y &= 3. \end{aligned}$$

The augmented matrix for this linear system is

$$\begin{pmatrix} i & 1 & 4 \\ -1 & 1 & 3 \end{pmatrix}$$

The three operations we can perform on augmented matrices to try to simplify them are:

- multiplying a row by a nonzero constant;
- interchanging two rows; and
- adding a constant multiple of one row to another row.

These are called *elementary row operations.* In this case, first we should multiply the first row by $-i$:

$$\begin{pmatrix} 1 & -i & -4i \\ -1 & 1 & 3 \end{pmatrix}$$

Then we should add the first row to the second row:

$$\begin{pmatrix} 1 & -i & -4i \\ 0 & 1-i & 3-4i \end{pmatrix}$$

This matrix corresponds to the system

$$\begin{aligned} x - iy &= -4i \\ (1-i)y &= 3-4i \end{aligned}$$

which, as before, can be solved by backsubstitution.