Practice Final 1

1. Answer the following questions "true," "false," or "I don't know." Five points will be given for the correct answer, and three points will be given for "I don't know."

   (a) The functions $f(x) = x$ and $g(x) = |x|$ are linearly independent on the interval $[-1, 1]$.

   (b) Let A be a matrix of rank 2 and B be a matrix of rank 1, both $n \times n$. Then $rk(AB) = 1$.

   (c) The function $f(x) = \ln |x|$ is piecewise continuous on $\mathbb{R}$.

   (d) Let P be an orthogonal projection matrix for a subspace V of $\mathbb{R}^3$. Then P is diagonalizable.

   (e) For any functions $p(x)$ and $q(x)$ which are continuous on $\mathbb{R}$, there exists a function $y(x)$ such that $y'' + py' + qy = 0$, $y(0) = 0$, and $y'(1) = 1$.

2. Consider the following system of differential equations.

$$\mathbf{x}'(t) = \frac{1}{3}\begin{pmatrix} -1 & -2 & -7 \\ -4 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}\mathbf{x}(t) \qquad (1)$$

   (a) Find the general solution to (1).

   (b) Find the solution to (1) such that

$$\mathbf{x}(0) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

3. Consider the partial differential equation

$$u_{xx} + u_{xt} + u_t = 0.$$

   (a) Transform this equation into a collection of ordinary differential equations using separation of variables.

   (b) Find all solutions to your ordinary differential equations from above such that $u(0, t) = u(\pi, t) = 0$.

4. Consider the following differential equation.

$$y''' - 3y' - 2y = 0 \qquad (2)$$

   (a) Write (2) as a system of first-order differential equations.

   (b) Give the general solution to (2).

   (c) Give the solution to (2) such that $y(4) = 2$, $y'(4) = 0$, and $y''(4) = -1$.

5. Consider the following linear transformation $T : P_2 \to P_2$.

$$T(a_0 + a_1x + a_2x^2) = a_0 + a_2 - a_1x + (a_0 + a_1 + a_2)x^2$$

   (a) Give a basis for the kernel of T.

   (b) Give the matrix for T relative to the basis $\{1, x - 1, (x - 1)^2\}$.

Practice Final 2

1. Answer the following questions "true," "false," or "I don't know." Five points will be given for the correct answer, and three points will be given for "I don't know."

   (a) The function

$$f(x) = \begin{cases} -1 & x < 0 \\ 8 & x = 0 \\ 1 & x > 0 \end{cases}$$

   is piecewise continuous on $\mathbb{R}$.

   (b) Let A be an $n \times n$ matrix whose only eigenvalue in $\mathbb{C}$ is 5. Then $rk(A) > rk(A - 5I)$.

   (c) The functions $f(x) = e^x \cos x$ and $g(x) = e^x \sin x$ are linearly dependent on the interval $[-2\pi, 2\pi]$.

   (d) Let A and B be two 2×2 matrices. Then $(AB)^T = A^T B^T$.

   (e) Let $p(x)$ and $q(x)$ be continuous on $\mathbb{R}$. Then there exists a unique solution to the differential equation $p(x)y'' + 3y' - q(x)y = 0$ on the interval $(-1, 1)$ such that $y(0) = 1$ and $y'(0) = -3$.

2. Consider the following system of differential equations.

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 1 & 0 \\ -2 & 0 & 0 \\ 1 & 2 & -1 \end{pmatrix}\mathbf{x}(t) \qquad (3)$$

   (a) Find the general solution to (3).

   (b) Give the solution to (3) satisfying the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

3. Transform the partial differential equation

$$u_{xt} - \alpha u_{tt} = 0$$

into a collection of ordinary differential equations using separation of variables.

4. Solve the following heat conduction problem involving a wire of length 10.

$$\begin{aligned} 9u_{xx} &= u_t \\ u(0,t) &= 10 \\ u(10,t) &= 20 \\ u(x,0) &= \sin\left(\frac{\pi x}{5}\right) \end{aligned}$$

5. For any $x \in [-\pi, \pi)$, let

$$f(x) = x - \cos x.$$

For $x \notin [-\pi, \pi)$, extend f so that it is periodic of period 2π. Give the Fourier series for f.

6. Define an inner product on P_2 by

$$\langle p, q \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2) \qquad (4)$$

for all $p, q, \in P_2$.

(a) Give an orthogonal basis for the subspace

$$S = \{p \in P_2 \mid p(1) = 0\}$$

with respect to the inner product (4).

(b) Let $T : P_2 \to P_2$ be orthogonal projection onto S with respect to the inner product (4). Give the matrix for T relative to the ordered basis $\{1, x-1, x^2-1\}$.