1. Let A be an $n \times n$ matrix with eigenvalue λ. For each of the following statements, write the word "true" or "false."

   (a) The matrix $2A$ has eigenvalue 2λ.
   True. Let $\mathbf{v}$ be an eigenvector with eigenvalue 2. Then $2A\mathbf{v} = 2\lambda\mathbf{v}$.

   (b) The matrix $A + 2I$ has eigenvalue $\lambda + 2$.
   True. Let $\mathbf{v}$ be an eigenvector with eigenvalue 2. Then $(A + 2I)\mathbf{v} = (\lambda + 2)\mathbf{v}$.

2. Give an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^{-1}$.

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

First we need to find the characteristic polynomial of A.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & -1 & -1 \\ -1 & \lambda - 4 & -1 \\ -1 & -1 & \lambda - 4 \end{vmatrix} = (\lambda - 3)^2(\lambda - 6)$$

To find the eigenspace for $\lambda = 6$, which must be one-dimensional, we need to look at

$$NS(A - 6I) = NS \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

which is spanned by $(1, 1, 1)^T$. Therefore, let $\mathbf{v}_1 = \frac{1}{\sqrt{3}}(1, 1, 1)^T$.

Next we need to find an orthonormal basis for the eigenspace for $\lambda = 3$, which must be two-dimensional by the Spectral Theorem. First we need a basis for

$$NS(A - 3I) = NS \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

which is spanned by $(1, -1, 0)^T$ and $(1/2, 1/2, -1)^T$, which are already orthogonal. Therefore let $\mathbf{v}_2 = \frac{1}{\sqrt{2}}(1, -1, 0)^T$ and $\mathbf{v}_3 = \frac{\sqrt{2}}{\sqrt{3}}(1/2, 1/2, -1)^T$.

Therefore, if

$$Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

then $A = Q\Lambda Q^{-1}$.