MATH 54 Lecture Notes 9

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July 18, 2006

1 Inner Products and Projections

1.1 Projection Matrices

Let P be a matrix that projects orthogonally onto a subspace V of $\mathbb{R}^n$. Then $P\mathbf{v} = \mathbf{v}$ for any $\mathbf{v} \in V$. Since the image of f_P is V, this implies that $P^2 = P$.

1.2 Cauchy-Schwarz

In any inner product space,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|.$$

Therefore, if we define $\cos\theta$ in contexts where it has no clear geometric meaning by means of the inner product, we know that $|\cos\theta| \leq 1$. A consequnce of Cauchy-Schwarz is the Triangle Inequality:

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

2 Adjoint Matrices

Let $\langle \, , \, \rangle$ denote the standard dot product on $\mathbb{C}^m$ or $\mathbb{C}^n$. Let A be an $m \times n$ matrix. Then the *adjoint* of A, written A^*, is given by

$$A^* = \overline{A^T}.$$

Here, complex conjugation of a matrix means taking the complex conjugate of each entry. (In particular, if A has all real entries, $A^* = A^T$.) For any vectors $\mathbf{u} \in \mathbb{C}^n$ and $\mathbf{v} \in \mathbb{C}^m$,

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A^*\mathbf{v} \rangle.$$

This can be used to prove that $CS(A^*A) = CS(A^*)$, from page 258.

A matrix A is called *self-adjoint* if $A = A^*$.

3 Least Squares

We can think of a traditional least squares problem in terms of vector spaces.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be a basis for a space in which we are trying to approximate $\mathbf{u}$. Let A be a matrix with columns equal to $\mathbf{v}_i$. Then we are trying to find $\mathbf{x}$ such that $A\mathbf{x}$ is close to $\mathbf{u}$. This happens when $\mathbf{u} - A\mathbf{x}$ is perpendicular to each $\mathbf{v}_i$. Therefore we want

$$A^* A\mathbf{x} = A^* \mathbf{u}.$$

This always has a solution.