MATH 54 Lecture Notes 8

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1 Linear Transformations

Let V and W be vector spaces. Then a function $T : V \to W$ is called a *linear transformation* if, for all $\mathbf{u}, \mathbf{v} \in V$ and all scalars α,

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$; and
- $T(\alpha\mathbf{u}) = \alpha T(\mathbf{u})$.

The most basic example is matrix multiplication. Let $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, and A be some $m \times n$ matrix. Then there is a linear transformation $f_A : V \to W$ given by $f_A(\mathbf{u}) = A\mathbf{u}$ for all $\mathbf{u} \in V$.

Note that it is enough to check that $T(\alpha\mathbf{u}+\mathbf{v}) = \alpha T(\mathbf{u})+T(\mathbf{v})$ for all scalars α and all vectors $\mathbf{u}$ and $\mathbf{v}$ in the domain of T. This test shows that the first rule is satisfied by setting $\alpha = 1$. Then since the first rule implies that $T(\mathbf{0}) = \mathbf{0}$, setting $\mathbf{v} = \mathbf{0}$ shows that this test implies the second rule.

Any linear transformation must send $\mathbf{0}$ to $\mathbf{0}$. This gives a quick way to check whether something is a linear transformation, such as in exercise 34.

The range of a linear transformation $T : V \to W$ is a subspace of W.

Any linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by some $m \times n$ matrix. (We can replace $\mathbb{R}^n$ and $\mathbb{R}^m$ with $\mathbb{C}^n$ and $\mathbb{C}^m$, and it's still true.) To find the i-th column of this matrix, just check where the transformation sends $\mathbf{e}_i$, such as in exercise 13.

Exercise 18. Linear.

Exercise 20. Not linear. Take $(x, y) = (0, 1)$ and check scalar multiplication.

Exercise 23. Linear. Any function which sends every vector in a vector space to $\mathbf{0}$ is always linear.

Exercise 36. Linear.

Exercise 37. Linear.

Exercise 39. Not linear. Consider $T(2I)$.

Exercise 40. Not linear. Consider $T(2I)$.

Exercise 41. For part (a), J is linear. For part (b), we are told the domain is $C[a,b]$. As long as $b > a$, the range is then all of $\mathbb{R}$.

Exponentiation as a linear map. Consider the strange vector space we had earlier, where the underlying set of vectors is $\mathbb{R}^+$, and the operations are defined by

$$\begin{aligned} x \oplus y &= xy \\ \alpha \otimes x &= x^\alpha \end{aligned}$$

for any $x, y \in \mathbb{R}^+$ and any $\alpha \in \mathbb{R}$. Then let $E : \mathbb{R} \to \mathbb{R}^+$ be given by $E(\alpha) = e^\alpha$. With the way things are set up, E is a linear transformation. (It has an inverse linear transformation. What is it? Can you prove that it's linear?)

2 Inner Product Spaces

2.1 Definition

Let V be a $\mathbb{C}$-vector space. Then an *inner product* on V is any function from ordered pairs in V to $\mathbb{C}$, $\langle\ ,\ \rangle$, with the following properties.

- Linear in the first variable. That is, for any $\alpha \in \mathbb{C}$ and any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

  $$\langle \alpha\mathbf{u}, \mathbf{w}\rangle = \alpha\langle \mathbf{u}, \mathbf{w}\rangle$$

  and

  $$\langle \mathbf{u} + \mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle.$$

- Conjugate-linear in the second variable. For any $\alpha \in \mathbb{C}$ and any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

  $$\langle \mathbf{u}, \alpha\mathbf{v}\rangle = \overline{\alpha}\langle \mathbf{u}, \mathbf{v}\rangle$$

  and

  $$\langle \mathbf{u}, \mathbf{v} + \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle.$$

  This property is actually redundant.

- For any $\mathbf{u}, \mathbf{v} \in V$,

  $$\langle \mathbf{u}, \mathbf{v}\rangle = \overline{\langle \mathbf{v}, \mathbf{u}\rangle}.$$

- For any $\mathbf{u} \in V$, $\langle \mathbf{u}, \mathbf{u}\rangle \geq 0$ (in particular, the inner product of any vector with itself is a real number), and $\langle \mathbf{u}, \mathbf{u}\rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

We can also have an inner product on an $\mathbb{R}$-vector space. In that case, the inner product of any two vectors should be a real number, and all above references to scalars apply only to real numbers. In this case, the above definition matches the one in the book.

2.2 Dot Product

The simplest example of an inner product is the dot product on $\mathbb{C}^n$. Given $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, the dot product of $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{n} a_i \overline{b_i}.$$

We can use this to define the length of a vector $\mathbf{u}$:

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}.$$

Note that the last axiom for inner products tells us that the length of any vector, when defined this way, is a nonnegative real number, and that only the zero vector can have length 0.

The length of $\mathbf{e}_i$ is 1.

Exercise 3.2.15. $\|(1, -2, 0, 2)\| = \sqrt{1 + 4 + 4} = 3.$