

# MATH 54 Lecture Notes 7

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## 1 Linear Independence

The vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$  are said to be *linearly independent* if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_r\mathbf{v}_r = \mathbf{0}$$

implies that

$$a_1 = a_2 = \dots = a_r = 0.$$

A set of vectors is said to be *linearly dependent* if they are not linearly independent.

Examples:

- $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  are linearly independent.
- Let  $\mathbf{e}_1 = (1, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, 0)$ , and  $\mathbf{u} = (1, 1, 0)$ . Then these three vectors are linearly dependent, since  $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{u} = \mathbf{0}$ .

In  $\mathbb{R}^n$ , any set of vectors with more than  $n$  elements is linearly dependent over  $\mathbb{R}$ . In  $\mathbb{C}^n$ , any set of vectors with more than  $n$  elements is linearly dependent over  $\mathbb{C}$ .

**Exercise 1.** Since  $3\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{0}$ , they are linearly dependent.

**Exercise 4.** Since  $2p_1 + p_2 = 0$ , they are linearly dependent.

**Exercise 6.** Since  $f_1(x) + f_2(x) - f_3(x) = 0$  for all  $x$ , they are linearly dependent.

**Exercise 8.** Any dependence relation corresponds to a solution to  $A\mathbf{x} = \mathbf{0}$ , where

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Since  $A$  is invertible, there is only the trivial solution, so  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent. If there had been other solutions,  $\mathbf{v}_1$  and  $\mathbf{v}_2$  would be linearly dependent.

**Exercise 28.** Suppose  $\sum_{i=1}^4 a_i \mathbf{u}_i = \mathbf{0}$ .

**Exercise 30.** Let  $\sum_{i=1}^n a_i \mathbf{u}_i = \mathbf{0}$  be a nontrivial dependence relation.

**Exercise 34.** Let's follow the hint...

## 2 Bases

Let  $V$  be a vector space, and let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r \in V$ . Then we say that these vectors are a *basis* for  $V$  if

- they are linearly independent, and
- they span  $V$  (i.e.  $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ ).

The vectors  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  form a basis for  $\mathbb{R}^3$  (or  $\mathbb{C}^3$ ). They are called the *standard basis*.

**Exercise 10.** For any  $\mathbf{b} \in \mathbb{R}^2$ , we want to know whether there exist  $a_1$  and  $a_2$  such that

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 = \mathbf{b}.$$

Equivalently, we can ask whether there is a solution to the matrix equation

$$\begin{pmatrix} 2 & 3 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \mathbf{b}.$$

There is a unique solution for any  $\mathbf{b}$ , since the matrix is invertible. This tells us that  $\mathbf{x}_1$  and  $\mathbf{x}_2$  span  $\mathbb{R}^2$ . The uniqueness of the solution in the case  $\mathbf{b} = \mathbf{0}$  tells us that they are linearly independent. Therefore they are a basis for  $\mathbb{R}^2$ .