MATH 54 Lecture Notes 4

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1 Vectors

- A *vector* in $\mathbb{R}^n$ or $\mathbb{C}^n$ is an ordered list of numbers. Vectors of the same size can be added together, and addition is done componentwise. Vectors can also be multiplied by scalars, and this mulitplication is done componentwise.
- Vectors can be thought of as arrows representing physical quantities such as displacement or force. In this case, the same vector can be drawn starting from any base point, and addition is done by connecting the arrows head to tail.
- Vectors also correspond to points. This is equivalent to think of vectors as arrows, except where the base point is always the origin.

2 Linear Combinations and Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$

Suppose we have vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. A *linear combination* of the $\mathbf{v}_i$'s refers to a vector of the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_r\mathbf{v}_r$$

for some scalars $a_1, a_2, \ldots, a_r$. The notation $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ refers to the set of all linear combinations of the $\mathbf{v}_i$'s. Some examples:

- $\mathbf{0}$ is always a linear combination of any set of vectors. It is obtained by letting all the scalars be equal to 0.
- $\mathbf{v}_1$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$.
- Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$. Then any vector with three entries is a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.
- Let $\mathbf{v} = (1, 2)$. What is $\text{Span}\{\mathbf{v}\}$?
- What is $\text{Span}\{\mathbf{0}\}$?

- Let $A\mathbf{x} = \mathbf{b}$ for some vectors $\mathbf{x}$ and $\mathbf{b}$ and some matrix A. Then $\mathbf{b}$ is a linear combination of the columns of A.

- Consider $\mathbf{e}_1$ and $\mathbf{e}_2$ as before, and $\mathbf{u} = (1, 1, 0)$. What is $\text{Span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{u}\}$?

In $\mathbb{R}^3$ (or $\mathbb{C}^3$), the span of any set of vectors is always either $\{\mathbf{0}\}$, a line passing through the origin, a plane passing through the origin, or all of $\mathbb{R}^3$ (or $\mathbb{C}^3$). The dimension of the span of a set of vectors is always less than or equal to the number of vectors in that set. When the dimension of the span and the number of vectors are equal, we say that the vectors are *linearly independent*.

Suppose we have two sets of vectors, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s\}$, and we wish to show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} \subseteq \text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s\}$. Then it is enough to show that $\mathbf{v}_i \in \text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s\}$ for each i. This can also be used to show that two spans are equal.

3 Axioms of Vector Spaces

For now, we refer to $\mathbb{R}^n$ or $\mathbb{C}^n$ as vector spaces (soon there will be others). The following axioms hold in a vector space, where r and s are scalars and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors:

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
- $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
- $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$.
- $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$.
- $(rs)\mathbf{u} = r(s\mathbf{u})$.
- $1 \cdot \mathbf{u} = \mathbf{u}$.