

MATH 54 Lecture Notes 19

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1 Euler's Formula and Magnitude

Let $\alpha, \beta \in \mathbb{R}$. Then

$$|e^{\alpha+\beta i}| = e^\alpha |\cos \beta + i \sin \beta| = e^\alpha \sqrt{\cos^2 \beta + \sin^2 \beta} = e^\alpha.$$

Also note that

$$e^{2\pi i} = \cos(2\pi) + i \sin(2\pi) = 1.$$

Therefore the exponential function is periodic with period $2\pi i$.

2 Fundamental Sets of Solutions

Consider a differential equation

$$\mathbf{x}'(t) = A\mathbf{x}(t). \tag{1}$$

Then we know that all solutions have the form

$$\mathbf{x}(t) = \exp(At)\mathbf{x}_0$$

for some vector \mathbf{x}_0 . Therefore the set of all solutions is simply $CS(\exp(At))$. However, the column space of a matrix is unaffected by multiplication by an invertible matrix on the right. If $A = SJS^{-1}$, where J is in Jordan canonical form, we already know that

$$\exp(At) = S \exp(Jt) S^{-1}.$$

Therefore

$$CS(\exp(At)) = CS(S \exp(Jt)).$$

If we want a basis for the set of all solutions to (1), then we can simply look at the columns of the matrix $S \exp(Jt)$.

3 Sources, Sinks, etc.

Consider a differential equation

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

where A is a diagonalizable 2×2 matrix with real entries. Several things can happen at the origin:

- Source. The eigenvalues of A are real and positive, so solutions move away from the origin as $t \rightarrow \infty$. Exercise 7.5.6.
- Sink. The eigenvalues of A are real and positive, so solutions move towards the origin as $t \rightarrow \infty$. Exercise 7.5.5. Do the umbrella demonstration here. The hanging umbrella is a sink; the balanced umbrella is a source.
- Saddle point. The eigenvalues of A are real with opposite signs. Exercise 7.5.3.
- Spiral in. The eigenvalues of A are complex with negative real part. Exercise 7.6.2.
- Spiral out. The eigenvalues of A are complex with positive real part. Exercise 7.6.4.
- Periodic. The eigenvalues of A are purely imaginary. Exercise 7.6.6.