MATH 54 Lecture Notes 18

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1 Initial Conditions

Consider the following initial value problem

$$\begin{aligned}\mathbf{x}'(t) &= A\mathbf{x}(t)\\ \mathbf{x}(t_0) &= \mathbf{x}_0\end{aligned}$$

where $t_0 \neq 0$. Let $\mathbf{y}(t) = \mathbf{x}(t + t_0)$. Then $\mathbf{y}(t - t_0) = \mathbf{x}(t)$, so $\mathbf{y}$ is a solution to the initial value problem

$$\begin{aligned}\mathbf{y}'(t - t_0) &= A\mathbf{y}(t - t_0)\\ \mathbf{y}(0) &= \mathbf{x}_0.\end{aligned}$$

So, when dealing with systems of differential equations of this form, we will assume that $t_0 = 0$.

As an example, consider the following initial value problem.

$$\begin{aligned}y'' + 6y' + 9y &= 0\\ y(4) &= 1\\ y'(4) &= 2\end{aligned}$$

Then if we set $z(t - 4) = y(t)$, then z is a solution to

$$\begin{aligned}z'' + 6z' + 9z &= 0\\ z(0) &= 1\\ z'(0) &= 2.\end{aligned}$$

The solution is $z = e^{-3t} + 5te^{-3t}$. Therefore the solution to our original initial value problem is $y = e^{-3(t-4)} + 5te^{-3(t-4)}$.

2 Matrix Exponentiation

2.1 Definition

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$. Then we define the exponential of A, denoted $\exp(A)$, to be the matrix

$$\sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2} + \frac{A^3}{6} + \frac{A^4}{24} + \cdots$$

This matrix will play the same role in solving homogeneous systems of first-order linear differential equations that the scalar exponential function serves in solving homogeneous higher-order linear differential equations. First we need to show some properties of the exponential of a matrix.

2.2 Basic Properties

The first properties we would like to show are ones that indicate the the matrix exponential behaves in a similar way to the scalar exponential function. For instance,

$$\exp(0) = I + 0 + \frac{0^2}{2} + \cdots = I.$$

So the exponential of the zero matrix is the identity matrix. Next, suppose A and B are two $n \times n$ matrices such that $AB = BA$. For any two such matrices, the binomial theorem holds. That is,

$$(A+B)^n = \sum_{j=0}^{n} \binom{n}{j} A^j B^{n-j}$$

where

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}.$$

Then

$$\begin{aligned} \exp(A+B) &= \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} A^j B^{n-j} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{A^j}{j!} \cdot \frac{B^{n-j}}{(n-j)!} \\ &= \exp(A)\exp(B). \end{aligned}$$

Since A always commutes with $-A$,

$$\exp(A)\exp(-A) = \exp(A - A) = \exp(0) = I.$$

Therefore the exponential of any matrix is always invertible, and $\exp(A)^{-1} = \exp(-A)$. Also, for any matrix A,

$$\begin{aligned} \frac{d}{dt}\exp(At) &= \frac{d}{dt}\sum_{n=0}^{\infty}\frac{A^n t^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{nA^n t^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} \\ &= A\sum_{n=1}^{\infty}\frac{A^{n-1}t^{n-1}}{(n-1)!} \\ &= A\exp(At). \end{aligned}$$

2.3 Exponentiation and Similarity

Now suppose $A = P^{-1}BP$, for some $n \times n$ matrices A, B, and P. We wish to relate $\exp(A)$, $\exp(B)$, and P. We have that

$$\exp(A) = \sum_{n=0}^{\infty}\frac{A^n}{n!} = \sum_{n=0}^{\infty}\frac{P^{-1}B^nP}{n!} = P^{-1}\left(\sum_{n=0}^{\infty}\frac{B^n}{n!}\right)P = P^{-1}\exp(B)P.$$

Therefore, if $A = SJS^{-1}$ where J is a block diagonal matrix (such as the Jordan canonical form of A), we can compute $\exp(A)$ using the formula

$$\exp(A) = S\exp(J)S^{-1}.$$

If

$$J = \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_r \end{pmatrix}$$

for some square matrices $B_1, B_2, \ldots, B_r$, then

$$J^n = \begin{pmatrix} B_1^n & & & \\ & B_2^n & & \\ & & \ddots & \\ & & & B_r^n \end{pmatrix}$$

so that

$$\begin{aligned}
\exp(J) &= \sum_{n=0}^{\infty} \frac{J^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} B_1^n & & & \\ & B_2^n & & \\ & & \ddots & \\ & & & B_r^n \end{pmatrix} \\
&= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{B_1^n}{n!} & & & \\ & \sum_{n=0}^{\infty} \frac{B_2^n}{n!} & & \\ & & \ddots & \\ & & & \sum_{n=0}^{\infty} \frac{B_r^n}{n!} \end{pmatrix} \\
&= \begin{pmatrix} \exp(B_1) & & & \\ & \exp(B_2) & & \\ & & \ddots & \\ & & & \exp(B_r) \end{pmatrix}.
\end{aligned}$$

So, if we want to calculate $\exp(At)$, it is enough to be able to calculate $\exp(Bt)$ where B is some Jordan block.

2.4 Computation of $\exp(At)$

2.4.1 The exponential of a 1×1 Jordan block

A 1×1 Jordan block is simply λI for some λ. In this case,

$$\exp(\lambda I t) = \sum_{n=0}^{\infty} \frac{\lambda^n I t^n}{n!} = \left(\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \right) I = e^{\lambda t} I. \tag{1}$$

2.4.2 The exponential of a 2×2 Jordan block

Define a matrix

$$N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then any 2×2 Jordan block B is equal to $\lambda I + N_2$ for some λ. Note that $\lambda I t$ and $N_2 t$ commute, since the former is a scalar multiple of I, so that $\exp(Bt) = \exp(\lambda I t + N_2 t) = \exp(\lambda I t)\exp(N_2 t)$. We already computed $\exp(\lambda I t) = e^{\lambda t} I$ in (1), so we only need to compute $\exp(N_2 t)$. Since $N_2^2 = 0$,

$$\exp(N_2 t) = \sum_{n=0}^{\infty} \frac{N_2^n t^n}{n!} = \sum_{n=0}^{1} \frac{N_2^n t^n}{n!} = I + N_2 t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

This gives us that

$$\exp(Bt) = \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

2.4.3 The exponential of a 3×3 Jordan block

Define a matrix

$$N_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

As above, any 3×3 Jordan block B is equal to $\lambda I + N_3$ for some λ. Also, $\lambda I t$ and $N_3 t$ commute, so that $\exp(Bt) = \exp(\lambda I t + N_3 t) = \exp(\lambda I t)\exp(N_3 t) = e^{\lambda t}\exp(N_3 t)$ by (1). Since $N_3^3 = 0$,

$$\exp(N_3 t) = \sum_{n=0}^{2} \frac{N_3^n t^n}{n!} = I + N_3 t + \frac{N_3^2 t^2}{2} = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$\exp(Bt) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2 e^{\lambda t}}{2} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

3 Systems of Homogeneous First-Order Linear Differential Equations with Constant Coefficients

Consider the following initial value problem, where A is some $n \times n$ matrix with constant coefficients.

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$
$$\mathbf{x}(0) = \mathbf{x}_0$$

The solution to this initial value problem is

$$\mathbf{x}(t) = \exp(At)\mathbf{x}_0.$$

This is a solution to the system of differential equations because $\frac{d}{dt}\exp(At) = A\exp(At)$, and it has the appropriate value at $t = 0$ because $\exp(0) = I$.

Boyce-DiPrima refers to the matrix $\exp(At)$ as the fundamental matrix of this system of differential equations.