MATH 54 Lecture Notes 17

GSI Carter

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1 Abel's Theorem

Consider a linear homogeneous differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0 \tag{1}$$

where each $p_i(t)$ is continuous on an open interval I containing a point t_0. Let this equation have solutions $y_1, y_2, \ldots, y_n$. These solutions may or may not be linearly independent. We wish to find a formula for $W = W(y_1, y_2, \ldots, y_n)$ which depends only on $p_1(t), p_2(t), \ldots, p_n(t)$ as much as possible. First we need to try to compute W'. We will use the following formula for the determinant:

$$\det A = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

where A is an $n \times n$ matrix with entries $a_{i,j}$. The $\sigma \in S_n$ and $(-1)^\sigma$ parts are probably nonsense[1] to you, but don't worry, because they're not that important. The point of the above formula is that the determinant can be written as a sum, where each term in the sum is a product of n entries in the matrix, each from a different row and different column. We will also use the following generalized product rule:

$$(f_1 f_2 \cdots f_n)' = f_1' f_2 \cdots f_n + f_1 f_2' \cdots f_n + \cdots + f_1 f_2 \cdots f_n'.$$

Using these two formulas, we obtain

$$\begin{aligned}
W' &= \sum_{\sigma \in S_n} (-1)^\sigma \left(y_{\sigma(1)} y'_{\sigma(2)} \cdots y^{(n-1)}_{\sigma(n)} \right)' \\
&= \sum_{\sigma \in S_n} (-1)^\sigma \left[y'_{\sigma(1)} y'_{\sigma(2)} \cdots y^{(n-1)}_{\sigma(n)} + y_{\sigma(1)} y''_{\sigma(2)} \cdots y^{(n-1)}_{\sigma(n)} + \cdots + y_{\sigma(1)} y'_{\sigma(2)} \cdots y^{(n)}_{\sigma(n)} \right] \\
&= \sum_{\sigma \in S_n} (-1)^\sigma y'_{\sigma(1)} y'_{\sigma(2)} \cdots y^{(n-1)}_{\sigma(n)} + \sum_{\sigma \in S_n} (-1)^\sigma y_{\sigma(1)} y''_{\sigma(2)} \cdots y^{(n-1)}_{\sigma(n)} + \cdots + \sum_{\sigma \in S_n} (-1)^\sigma y_{\sigma(1)} y'_{\sigma(2)} \cdots y^{(n)}_{\sigma(n)} \\
&= \begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \cdots + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} \\
&= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}.
\end{aligned}$$

[1] This footnote is only present for completeness' sake, and in case anyone who plans on being a math major wants to read it. Here, S_n refers to the set of all permutations of the set $\{1, 2, \ldots, n\}$. Each $\sigma(i)$ in the subscript denotes the i-th element in that permutation. The quantity $(-1)^\sigma$ denotes the sign of the permutation. It is 1 for permutations that are the product of an even number of transpositions and -1 for permutations that are the product of an odd number of transpositions. This topic is covered in MATH 113.

Using row reduction operations which preserve the determinant, and the fact that each y_i is a solution to (1), we obtain

$$W' = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ -p_1(t)y_1^{(n-1)} & -p_1(t)y_2^{(n-1)} & \cdots & -p_1(t)y_n^{(n-1)} \end{vmatrix} = -p_1(t)W.$$

Then $\frac{dW}{dt} + p_1(t)W = 0$. This equation is separable, so we have $\frac{dW}{W} = -p_1(t)\,dt$. Therefore

$$W(t) = ce^{-\int_{t_0}^{t} p_1(x)\,dx} \tag{2}$$

for some c which is constant with respect to t. Equation (2) is called Abel's Theorem. It is Theorem 3.3.2 and Exercise 4.1.20 in Boyce-DiPrima. Since $e^{-\int_{t_0}^{t} p_1(x)\,dx} \neq 0$ for all $t \in I$ and c does not depend on t, this implies that $W(t)$ is either zero for all $t \in I$ or nonzero for all $t \in I$.

To summarize, if $y_1, y_2, \ldots, y_n$ are solutions to (1), then the following are equivalent:

1. The functions $y_1, y_2, \ldots, y_n$ span the solution set of (1).[2]
2. The functions $y_1, y_2, \ldots, y_n$ are linearly equivalent over I.
3. The Wronskian $W(y_1, y_2, \ldots, y_n)(t) \neq 0$ for some $t \in I$.
4. The Wronskian $W(y_1, y_2, \ldots, y_n)(t) \neq 0$ for all $t \in I$.

Exercise 3.3.22. A linearly independent set of solutions must have a nonzero Wronskian. This implies that $p(t) = 0$ for all $t \in I$.

Exercise 3.3.24. The Wronskian $W(y_1, y_2)$ would have to be zero at such a point, so it would be zero over all of I.

Exercise 3.3.28. On $(0, 1)$, $f = g$. On $(-1, 0)$, $f = -g$. However, on $(-1, 1)$, consider the linear map

$$T(h) = \begin{pmatrix} h(1/2) \\ h(-1/2) \end{pmatrix}.$$

The map T sends f and g to a basis for $\mathbb{R}^2$, so they are linearly independent. However, their Wronskian is zero.

2 Systems of Linear First-Order Differential Equations

2.1 Existence and Uniqueness

Let I be an open interval containing a point t_0. Let $A(t)$ be an $n \times n$ matrix, where each entry is a function of t which is continuous on I. Also let $\mathbf{g}(t)$ be an n-element vector, where each entry is also a function of t which is continuous on I. Then a matrix equation

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{g}(t) \tag{3}$$

is what we call a system of linear first-order differential equations. Here, taking the derivative of a vector simply means taking the derivative of each entry. A solution to (3) is an n-element vector $\mathbf{x}(t)$ where each entry is a differentiable function of t and (3) holds for all $t \in I$. Under these conditions, for any $\mathbf{x}_0 \in \mathbb{C}^n$, there exists a unique solution to (3) such that $\mathbf{x}(t_0) = \mathbf{x}_0$. This is Theorem 7.1.2 in Boyce-DiPrima.

We can rewrite (3) as

$$\mathbf{x}'(t) - A(t)\mathbf{x}(t) = \mathbf{g}(t).$$

This way, the left hand side of the equation is a linear transformation of $\mathbf{x}(t)$. Then, as with a single differential equation, the corresponding homogeneous system of equations is

$$\mathbf{x}'(t) - A(t)\mathbf{x}(t) = \mathbf{0} \tag{4}$$

and the set of all solutions to (3) is obtained by taking any particular solution and then adding all solutions to (4).

[2] Boyce-DiPrima refers to this as $y_1, y_2, \ldots, y_n$ being a fundamental set of solutions for (1).

2.2 Higher-Order Linear Differential Equations

Suppose we have a linear differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = g(t).$$

We can turn this into a system of first-order linear differential equations by introducing new variables. For each $1 \leq i < n$, let $u_i = y^{(i)}$. Then we have the following first-order linear differential equations:

$$\begin{aligned} y' &= u_1 \\ u_1' &= u_2 \\ u_2' &= u_3 \\ &\vdots \\ u_{n-2}' &= u_{n-1} \\ u_{n-1}' &= g(t) - p_1(t)u_{n-1} - p_2(t)u_{n-2} - \cdots - p_n(t)y \end{aligned}$$

If we let

$$\mathbf{x} = \begin{pmatrix} y \\ u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{pmatrix}, \mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ g(t) \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p_n(t) & -p_{n-1}(t) & -p_{n-2}(t) & \cdots & -p_1(t) \end{pmatrix}$$

then the above system becomes $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$.

For an example of this, take the differential equation

$$y'' + ay' + by = 0.$$

Then we will set $u = y'$, so that $y' = u$ and $u' = -au - by$ is our system of differential equations. This is equivalent to

$$\begin{pmatrix} y' \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$\begin{vmatrix} \lambda & -1 \\ b & \lambda + a \end{vmatrix} = \lambda^2 + a\lambda + b$$

which is the same characteristic equation we had been using for the second-order equation.