MATH 54 Lecture Notes 15

GSI Carter

July 31, 2006

1 Euler's Formula

Recall the Taylor series expansions for e^x, $\sin x$, and $\cos x$ centered at $x = 0$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

We will use these to define $e^{i\theta}$ for any $\theta \in \mathbb{R}$.

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} \\
&= \cos\theta + i \sin\theta
\end{aligned}$$

This is called Euler's Formula. We can use it to exponentiate any complex number $z = a + bi$, where $a, b \in \mathbb{R}$.

$$e^z = e^{a+bi} = e^a e^{bi} = e^a(\cos b + i \sin b)$$

2 Linear Homogeneous Second-Order Differential Equations with Constant Coefficients

The first differential equation we will solve has the following form:

$$y'' + ay' + by = 0. \tag{1}$$

We wish to find all solutions. The first step is to compute what is called the characteristic equation of the differential equation. This equation is

$$\lambda^2 + a\lambda + b = 0.$$

How we solve the differential equation will depend on what kinds of roots the characteristic equation has.

2.1 Distinct Real Roots

Suppose $\lambda^2 + a\lambda + b = (\lambda - r_1)(\lambda - r_2)$ for some $r_1, r_2 \in \mathbb{R}$ with $r_1 \neq r_2$. Then the general solution to the differential equation (1) has the form

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Example. Solve $y'' - 2y' - y = 0$.

2.2 Repeated Real Root

Suppose $\lambda^2 + a\lambda + b = (\lambda - r)^2$ for some $r \in \mathbb{R}$. Then the general solution to (1) has the form

$$y = c_1 e^{rt} + c_2 t e^{rt}.$$

Example. Solve $y'' - 2y' + y = 0$.

2.3 Distinct Complex Roots

Since we are supposing that $a, b \in \mathbb{R}$, anytime there are complex roots, they must be distinct. This is because complex roots of polynomials with real coeffiicents occur in complex conjugate pairs. Suppose $\lambda^2 + a\lambda + b = (\lambda - \alpha - \beta i)(\lambda - \alpha + \beta i)$ for some $\alpha, \beta \in \mathbb{R}$. Then, as with distinct real roots, the general solution to (1) takes the form

$$y = d_1 e^{(\alpha+\beta i)t} + d_2 e^{(\alpha-\beta i)t}.$$

However, we started with a differential equation with real coefficients, and the variable t is a real-valued variable, so we would like to know which of these solutions are real valued functions. Note that the set of solutions is a two-dimensional $\mathbb{C}$-vector space spanned by

$$y_1 = e^{(\alpha+\beta i)t} = e^{\alpha t}\left[\cos(\beta t) + i\sin(\beta t)\right]$$

and

$$y_2 = e^{(\alpha-\beta i)t} = e^{\alpha t}\left[\cos(\beta t) - i\sin(\beta t)\right].$$

Another basis for the same $\mathbb{C}$-vector space is $\frac{y_1+y_2}{2}$ and $\frac{y_1-y_2}{2i}$. So,

$$\frac{y_1+y_2}{2} = \frac{e^{\alpha t}\left[\cos(\beta t) + i\sin(\beta t) + \cos(\beta t) - i\sin(\beta t)\right]}{2} = e^{\alpha t}\cos(\beta t)$$

and

$$\frac{y_1 - y_2}{2i} = \frac{e^{\alpha t}\left[\cos(\beta t) + i\sin(\beta t) - \cos(\beta t) + i\sin(\beta t)\right]}{2i} = e^{\alpha t}\sin(\beta t).$$

Therefore the set of all solutions to (1) can be written as

$$y = e^{\alpha t}\left[c_1\cos(\beta t) + c_2\sin(\beta t)\right].$$

Example. Solve $y'' - 2y' + 2y = 0$.