MATH 54 Lecture Notes 14: The Jordan Canonical Form

GSI Carter

July 27, 2006

1 Definition and Basic Facts

A *Jordan block* is a square matrix of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

That is, every diagonal entry is λ, for some $\lambda \in \mathbb{C}$, and every entry just above the main diagonal is 1. All other entries are 0. The following 1×1 matrix is also said to be a Jordan block:

$$(\lambda)$$

The *Jordan canonical form* of a square matrix A is a matrix of the form

$$J = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_r \end{pmatrix}$$

where each B_i is a Jordan block, and J is similar to A. We refer to such a matrix equation $A = SJS^{-1}$ as putting A in Jordan canonical form. The Jordan canonical form of any matrix is unique, up to rearranging the Jordan blocks. For any diagonalizable matrix, its Jordan canonical form is its diagonalization. In this case, each Jordan block is of size 1×1.

Putting a matrix in Jordan canonical form allows us to easily raise it to large powers, even if the matrix is not diagonalizable. For instance, if $A = SJS^{-1}$ with J as above, then

$$A^n = SJ^nS^{-1} = S \begin{pmatrix} B_1^n & 0 & \cdots & 0 \\ 0 & B_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_r^n \end{pmatrix} S^{-1}.$$

2 The Jordan Canonical Form of a 2×2 Matrix

Let A be a 2×2 matrix. If A is diagonalizable, then putting it into Jordan canonical form is the same as diagonalizing it. Suppose A is not diagonalizable. Then $\chi_A(\lambda) = (\lambda - \lambda_0)^2$ for some $\lambda_0 \in \mathbb{C}$, and $\dim NS(A - \lambda_0 I) = 1$. Since A is not diagonalizable, the only other possibility is that A has Jordan canonical form

$$J = \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix},$$

and we with to find a change of basis matrix S such that $A = SJS^{-1}$. Let $\mathbf{v}_2 \in \mathbb{C}^2$ be such that $(A - \lambda_0 I)\mathbf{v}_2 \neq \mathbf{0}$, which must exist since $\dim NS(A - \lambda_0 I) < 2$. Since

$$(A - \lambda_0 I)^2 = (S(J - \lambda_0 I)S^{-1})^2 = S(J - \lambda_0 I)^2 S^{-1} = 0,$$

$\mathbf{v}_2 \in NS(A - \lambda_0 I)^2$. If we let $\mathbf{v}_1 = (A - \lambda_0 I)\mathbf{v}_2$, then $\mathbf{v}_1$ is an eigenvector of A. Then we have that $A\mathbf{v}_1 = \lambda_0 \mathbf{v}_1$ and

$$A\mathbf{v}_2 = \mathbf{v}_1 + \lambda_0 \mathbf{v}_2.$$

Therefore, A relative to the ordered basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the matrix J. So, if S is the matrix with first column $\mathbf{v}_1$ and second column $\mathbf{v}_2$, then $A = SJS^{-1}$.

3 The Jordan Canonical Form of a 3×3 Matrix

3.1 With Three Distinct Eigenvalues

Any such matrix is diagonalizable. Therefore, the Jordan canonical form is the diagonalization.

3.2 With Two Distinct Eigenvalues

Let A be such a matrix which is not diagonalizable. Then $\chi_A(\lambda) = (\lambda - \lambda_0)^2(\lambda - \lambda_1)$ for some $\lambda_0, \lambda_1 \in \mathbb{C}$ with $\lambda_0 \neq \lambda_1$, and $\dim NS(A - \lambda_0 I) = \dim NS(A - \lambda_1 I) = 1$. The Jordan canonical form of A must be

$$J = \begin{pmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}.$$

Since

$$\dim NS(A - \lambda_0 I)^2 = \dim NS(S(J - \lambda_0 I)^2 S^{-1}) = \dim NS(J - \lambda_0 I)^2 = 2,$$

there exists $\mathbf{v}_2 \in \mathbb{C}^2$ such that $\mathbf{v}_2 \in NS(A - \lambda_0 I)^2$ but $\mathbf{v}_2 \notin NS(A - \lambda_0 I)$. Then if we set $\mathbf{v}_1 = (A - \lambda_0 I)\mathbf{v}_2$, the vector $\mathbf{v}_1$ is an eigenvector of A with eigenvalue λ_0. Finally, let $\mathbf{v}_3$ be an eigenvector of A with eigenvalue λ_1. Now we have that $A\mathbf{v}_1 = \lambda_0 \mathbf{v}_1$, $A\mathbf{v}_2 = \mathbf{v}_1 + \lambda_0 \mathbf{v}_2$, and $A\mathbf{v}_3 = \lambda_1 \mathbf{v}_3$. Then if S is the matrix with columns $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, we have that $A = SJS^{-1}$.

3.3 With a Unique Eigenvalue

Let A be such a matrix that is not diagonalizable. Then $\chi_A(\lambda) = (\lambda - \lambda_0)^3$ for some $\lambda_0 \in \mathbb{C}$, and $\dim NS(A - \lambda_0 I) < 3$. Therefore there are two possibilities to consider. In each case, the Jordan canonical form of A is an upper triangular matrix with only λ_0 on the diagonal, and either one or two entries equal to 1 directly above the diagonal. We will use the fact that $\dim NS(A - \lambda_0 I) = \dim NS(J - \lambda_0 I)$, where J is the Jordan canonical form of A.

3.3.1 $\dim NS(A - \lambda_0 I) = 2$

We know that

$$J = \begin{pmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix}.$$

Since $(A - \lambda_0 I)^2 = S(J - \lambda_0 I)^2 S^{-1} = 0$, there exists $\mathbf{v}_2 \in NS(A - \lambda_0 I)^2 = \mathbb{C}^3$ such that $\mathbf{v}_2 \notin NS(A - \lambda_0 I)$. Once this $\mathbf{v}_2$ is chosen, let $\mathbf{v}_1 = (A - \lambda_0 I)\mathbf{v}_2$, and let $\mathbf{v}_3$ be an eigenvector of A which is not a scalar multiple of $\mathbf{v}_1$. Then $A\mathbf{v}_1 = \lambda_0 \mathbf{v}_1$, $A\mathbf{v}_2 = \mathbf{v}_1 + \lambda_0 \mathbf{v}_2$, and $A\mathbf{v}_3 = \lambda_0 \mathbf{v}_3$. Therefore, if S has columns $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, then $A = SJS^{-1}$.

3.3.2 $\dim NS(A - \lambda_0 I) = 1$

We know that

$$J = \begin{pmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{pmatrix}.$$

Then $(A - \lambda_0 I)^3 = S(J - \lambda_0 I)^3 S^{-1} = 0$. Since

$$(J - \lambda_0 I)^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which has rank 1, $\dim NS(A - \lambda_0 I)^2 = 2$. Therefore there exists $\mathbf{v}_3 \in NS(A - \lambda_0 I)^3 = \mathbb{C}^3$ such that $\mathbf{v}_3 \notin NS(A - \lambda_0 I)^2$. If we let $\mathbf{v}_2 = (A - \lambda_0 I)\mathbf{v}_3$ and $\mathbf{v}_1 = (A - \lambda_0 I)\mathbf{v}_2 = (A - \lambda_0 I)^2 \mathbf{v}_3$, then we have that $A\mathbf{v}_1 = \lambda_0 \mathbf{v}_1$, $A\mathbf{v}_2 = \mathbf{v}_1 + \lambda_0 \mathbf{v}_2$, and $A\mathbf{v}_3 = \mathbf{v}_2 + \lambda_0 \mathbf{v}_3$. We can also see that these three vectors a linearly independent by looking at which null spaces they are contained in. Therefore, if S is the matrix with columns $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, then $A = SJS^{-1}$.