MATH 54 Lecture Notes 12

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1 Eigenvalues and Eigenvectors

1.1 Definitions

Let A be an $n \times n$ matrix. Then an *eigenvalue* for A is a scalar $\lambda \in \mathbb{C}$ such that there exists a nonzero vector $\mathbf{v} \in \mathbb{C}^n$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Any such vector $\mathbf{v}$ is called an *eigenvector*. We require $\mathbf{v}$ to be nonzero because otherwise every scalar is an eigenvalue. The above equation is equivalent to

$$(A - \lambda I)\mathbf{v} = A\mathbf{v} - \lambda\mathbf{v} = 0.$$

So we are looking for values of λ such that $NS(A - \lambda I)$ is nontrivial. This is equivalent to saying that the matrix $A - \lambda I$ is singular, which is in turn equivalent to

$$\det(\lambda I - A) = 0.$$

The expression $\det(\lambda I - A)$ is a polynomial of degree n in the variable λ. It is called the characteristic polynomial of A, which is written $\chi_A(\lambda)$. Thus, for any matrix A, the eigenvalues of A are the roots of the polynomial $\chi_A(\lambda)$.

Since the set of eigenvectors for a given eigenvector λ is equal to $NS(\lambda I - A)$, it is a subspace of $\mathbb{C}^n$. This subspace is called the eigenspace associated to λ.

1.2 Examples

It's easy to read off the eigenvalues of a diagonal matrix. Take, for example,

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then $\chi_A(\lambda) = (\lambda - 2)(\lambda + 1)(\lambda + 3)$. The eigenspaces are each one-dimensional.

It's also easy to compute the eigenvalues of upper-triangular matrices. Take the matrix

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\chi_B(\lambda) = (\lambda - 2)^2(\lambda - 1)$. In this case, there are two eigenspaces, each of dimension 1. Any $n \times n$ matrix has at most n linearly independent eigenvectors, but here's an example of a 3×3 matrix with only two linearly independent eigenvectors.

Exercise 13.

2 Traces and Determinants

The *trace* of an $n \times n$ matrix A is the sum of its diagonal entries.

The sum of the eigenvalues of any matrix is equal to its trace, and the product of the eigenvalues of any matrix is equal to its determinant. These sums and products must be computed with multiplicity, as illustrated by the above matrix B.