MATH 54 Lecture Notes 10

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1 Orthogonal Matrices

A collection of vectors is called *orthonormal* if the vectors are orthogonal and each of them has length 1. We can get an orthonormal basis from an orthogonal basis by dividing each vector by its length. For instance, take the orthogonal basis $(2, 1), (-1, 2)$ of $\mathbb{R}^2$.

A matrix is called *orthogonal* if it is square and its columns are orthonormal. Note the discrepancy of terminology! The discrepancy exists because the idea of a matrix with orthogonal columns is less useful.

For any orthogonal matrix Q, $Q^* = Q^{-1}$.

2 Similarity of Matrices

Let A and B be $n \times n$ matrices such that there exists an invertible $n \times n$ matrix P such that

$$A = P^{-1}BP.$$

Then we say that A and B are *similar*.

Exercise 31. Since $A = I^{-1}AI$, A is similar to itself.

Exercise 32. If $A = P^{-1}DP$, then $D = PAP^{-1} = (P^{-1})^{-1}AP^{-1}$.

Exercise 33. If $A = P^{-1}DP$ and $D = Q^{-1}EQ$, then

$$A = P^{-1}(Q^{-1}EQ)P = (QP)^{-1}E(QP).$$

Therefore A is similar to E.

These three exercises show that similarity is an equivalence relation. This means that the relation of similarity divides the set of all matrices into disjoint classes of matrices, where all the matrices in each class are similar to each other.

Exercise 34. The identity matrix is only similar to itself.

Exercise 35. If $A = P^{-1}BP$,

$$A^2 = P^{-1}BPP^{-1}BP = P^{-1}B^2P.$$

The same argument shows that A^k is similar to B^k for all positive integers k.