1. Find the general solution to the following differential equation.

$$\mathbf{x}'(t) = \begin{pmatrix} -1 & 1 \\ -1 & -3 \end{pmatrix} \mathbf{x}(t)$$

Call the given matrix A. Then the characteristic polynomial of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & -1 \\ 1 & \lambda+3 \end{vmatrix} = \lambda^2 + 4\lambda + 3 + 1 = (\lambda+2)^2.$$

Then the only eigenvalue is -2, and the eigenspace for this eigenvalue is

$$NS(A+2I) = NS\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

If we let $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$$\mathbf{v}_1 = (A+2I)\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

then $\mathbf{v}_1$ is an eigenvector, and our two linearly independent solutions have the form $te^{-2t}\mathbf{v}_1 + e^{-2t}\mathbf{v}_2$ and $e^{-2t}\mathbf{v}_1$. Therefore the general solution is

$$\mathbf{x}(t) = c_1 e^{-2t}\mathbf{v}_1 + c_2 t e^{-2t}\mathbf{v}_1 + c_2 e^{-2t}\mathbf{v}_2 = e^{-2t}\left[c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} t+1 \\ -t \end{pmatrix} \right].$$

2. For $x \in (-\pi, \pi]$, define $f(x)$ by

$$f(x) = \begin{cases} 1 & -\pi < x \le 0 \\ 1-x & 0 < x \le \pi \end{cases}$$

and let $f(x) = f(x+2\pi)$ for all $x \in \mathbb{R}$. Find the Fourier series for $f(x)$.

We can write $f(x) = 1 - g(x)$, where

$$g(x) = \begin{cases} 0 & -\pi < x \le 0 \\ x & 0 < x \le \pi. \end{cases}$$

Since 1 is already a Fourier series, we only need to find the Fourier series for $g(x)$. Then we have

$$a_0 = \frac{1}{\pi}\int_0^\pi x\, dx = \frac{\pi}{2}.$$

For all $m > 0$,

$$\begin{aligned} a_m &= \frac{1}{\pi}\int_0^\pi x\cos(mx)\, dx \\ &= \frac{1}{\pi}\left[\frac{x\sin(mx)}{m} + \frac{\cos(mx)}{m^2} \right]\Bigg|_{x=0}^{\pi} \\ &= \frac{(-1)^m - 1}{\pi m^2} \end{aligned}$$

and

$$\begin{aligned}
b_m &= \frac{1}{\pi} \int_0^{\pi} x \sin(mx)\; dx \\
&= \frac{1}{\pi} \left[-\frac{x \cos(mx)}{m} + \frac{\sin(mx)}{m^2} \right] \Bigg|_{x=0}^{\pi} \\
&= \frac{(-1)^{m+1}}{m}.
\end{aligned}$$

Therefore

$$g(x) = \frac{\pi}{4} + \sum_{m=1}^{\infty} \left[\frac{(-1)^m - 1}{\pi m^2} \cos(mx) + \frac{(-1)^{m+1}}{m} \sin(mx) \right]$$

and hence

$$f(x) = 1 - \frac{\pi}{4} - \sum_{m=1}^{\infty} \left[\frac{(-1)^m - 1}{\pi m^2} \cos(mx) + \frac{(-1)^{m+1}}{m} \sin(mx) \right].$$