1. (a) Find the general solution to the following differential equation.

$$\mathbf{x}' = \frac{1}{3}\begin{pmatrix} 7 & 1 \\ 2 & 8 \end{pmatrix}\mathbf{x}$$

First we need the characteristic polynomial of the given matrix, which we call A.

$$\det(\lambda I - A) = \frac{1}{9}\begin{vmatrix} 3\lambda - 7 & -1 \\ -2 & 3\lambda - 8 \end{vmatrix} = \frac{1}{9}(9\lambda^2 - 45\lambda + 56 - 2) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

For $\lambda = 2$, the eigenspace is

$$NS(A - 2I) = NS\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\},$$

and for $\lambda = 3$ it is

$$NS(A - 3I) = NS\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}.$$

Therefore the general solution is given by

$$\mathbf{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(b) Which type of equilibrium point is the origin for the above differential equation? Circle one.

i. source
ii. sink
iii. saddle point

It is a source, since both eigenvalues are positive.

2. Find the general solution to the following differential equation.

$$\mathbf{x}' = \begin{pmatrix} -2 & 2 \\ -5 & 4 \end{pmatrix}\mathbf{x}$$

As above, we first need the characteristic polynomial, which we call B.

$$\det(\lambda I - B) = \begin{vmatrix} \lambda + 2 & -2 \\ 5 & \lambda - 4 \end{vmatrix} = \lambda^2 - 2\lambda - 8 + 10 = (\lambda - 1 - i)(\lambda - 1 + i).$$

For $\lambda = 1 + i$, the eigenspace is

$$NS(B - I - iI) = NS\begin{pmatrix} -3 - i & 2 \\ -5 & 3 - i \end{pmatrix} = \operatorname{Span}\left\{\begin{pmatrix} 2 \\ 3 + i \end{pmatrix}\right\}.$$

Thus, one complex-valued solution is

$$\mathbf{x}_1(t) = e^{(1+i)t}\begin{pmatrix} 2 \\ 3 + i \end{pmatrix} = e^t[\cos t + i\sin t]\begin{pmatrix} 2 \\ 3 + i \end{pmatrix} = e^t\left[\begin{pmatrix} 2\cos t \\ 3\cos t - \sin t \end{pmatrix} + i\begin{pmatrix} 2\sin t \\ 3\sin t + \cos t \end{pmatrix}\right].$$

Therefore the general solution is

$$\mathbf{x}(t) = e^t\left[c_1\begin{pmatrix} 2\cos t \\ 3\cos t - \sin t \end{pmatrix} + c_2\begin{pmatrix} 2\sin t \\ 3\sin t + \cos t \end{pmatrix}\right].$$