1. Find all eigenvalues and eigenvectors for the following matrix.

$$A = \begin{pmatrix} -4 & -6 & 3 \\ 2 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

The characteristic polynomial is given by

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+4 & 6 & -3 \\ -2 & \lambda-4 & 2 \\ 2 & 2 & \lambda-1 \end{vmatrix} = \lambda^3 + \lambda^2 - 2\lambda = \lambda(\lambda+1)(\lambda-2).$$

The eigenvectors for $\lambda = 0$ are the null space of

$$\begin{pmatrix} -4 & -6 & 3 \\ 2 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so that a basis for the eigenspace is $(0,1,2)^T$.

For $\lambda = -1$, the eigenvectors are the null space of

$$\begin{pmatrix} -3 & -6 & 3 \\ 2 & 5 & -2 \\ -2 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that a basis for the eigenspace is $(1,0,1)^T$.

For $\lambda = 2$, the eigenvectors are the null space of

$$\begin{pmatrix} -6 & -6 & 3 \\ 2 & 2 & -2 \\ -2 & -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so that a basis for the eigenspace is $(1,-1,0)^T$.

2. Find all eigenvalues and eigenvectors for the following matrix.

$$B = \frac{1}{3}\begin{pmatrix} 4 & -1 & -1 \\ 0 & 6 & 0 \\ -2 & -10 & 5 \end{pmatrix}$$

The characteristic polynomial for $3B$ is given by

$$\begin{aligned} \det(\lambda I - 3B) &= \begin{vmatrix} \lambda-4 & 1 & 1 \\ 0 & \lambda-6 & 0 \\ 2 & 10 & \lambda-5 \end{vmatrix} \\ &= (\lambda-6)\left[(\lambda-4)(\lambda-5) - 2\right] \\ &= (\lambda-6)(\lambda^2 - 9\lambda + 18) \\ &= (\lambda-6)^2(\lambda-3). \end{aligned}$$

Therefore the eigenvalues of B are 1 and 2.

For $\lambda = 1$, the eigenvectors are the null space of

$$3B - 3I = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 3 & 0 \\ -2 & -10 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that a basis for the eigenspace is $(1, 0, 1)^T$.
For $\lambda = 2$, the eigenvectors are the null space of

$$3B - 6I = \begin{pmatrix} -2 & -1 & -1 \\ 0 & 0 & 0 \\ -2 & -10 & -1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that a basis for the eigenspace is $(-1, 0, 2)^T$.