MATH 54 Midterm 1 Review

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1 Linear Equations and Matrices

1.1 Things to Know

- How to row-reduce a matrix, and how to read the solution set of a linear system from its row-reduced matrix.
- If A is $m \times n$ and B is $n \times q$, then $C = AB$ is $m \times q$, and $C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$.
- In general $AB \neq BA$. However, $A(BC) = (AB)C$, and $A(B + C) = AB + AC$.
- If A is a square matrix, then the inverse A^{-1} is a matrix such that $A^{-1}A = AA^{-1} = I$, if such a matrix exists. In this case, A is called invertible.
- The inverse of a matrix A is unique if it exists.
- If A and B are both invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.
- How to compute inverses via row-reduction.
- The three types of elementary matrices.
- Let A be an $n \times n$ matrix. Then the following are equivalent:
  - A is invertible.
  - $AX = B$ has a unique solution for any B.
  - $AX = \mathbf{0}$ has only the trivial solution $X = \mathbf{0}$.
  - A is row equivalent to I_n.
  - A is a product of elementary matrices.

1.2 Problems

- Chapter 1 Review: 4, 6, 7, 12, 14, 15, 16.

2 Vector Spaces

2.1 Things to Know

- Simple formula for the dot product: $(a, b, c) \cdot (d, e, f) = ad + be + cf$.
- $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$.
- Two vectors are perpendicular if and only if their dot product is zero.
- $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
- The projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$. Note that any projection onto $\mathbf{v}$ is always a scalar multiple of $\mathbf{v}$.
- The Gram-Schmidt process.
- Axioms for a real vector space, for any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and any real numbers r and s:
  - Closed under addition.
  - Closed under scalar multiplication.
  - $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
  - $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
  - There exists a vector $\mathbf{0}$ such that, for any vector $\mathbf{u}$, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
  - There exists $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
  - $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$.
  - $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$.
  - $(rs)\mathbf{u} = r(s\mathbf{u})$.
  - $1 \cdot \mathbf{u} = \mathbf{u}$.
- Things which follow from the axioms:
  - $0 \cdot \mathbf{u} = \mathbf{0}$.
  - $r\mathbf{0} = \mathbf{0}$.
  - $(-1)\mathbf{u} = -\mathbf{u}$.

  - If $r\mathbf{u} = \mathbf{0}$, then either $r = 0$ or $\mathbf{u} = \mathbf{0}$.

- The set of $m \times n$ matrices is a vector space.
- A vector space contained within another vector space (and using the same operations as the bigger vector space) is called a subspace.
- To check if a subset of a vector space is a subspace, you only need to check that it is closed under addition and closed under scalar multiplication.
- The null space of a matrix A is the set of vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. It is always a vector space.
- Definitions of linear combination, span, and linear dependence.
- In $\mathbb{R}^n$, any set of more than n vectors must be linearly dependent.

2.2 Problems

- Chapter 3 Review: 2, 3, 4, 6, 7, 9, 11.
- Chapter 3 Cumulative Review: 1, 6, 7.
- Suppose p, q, r, and s are polynomials of degree 3 or less.
  - Suppose that $p(1) = q(1) = r(1) = s(1) = 0$. Is it possible for these polynomials to be linearly independent?
  - Now, instead suppose that $p(0) = q(0) = r(0) = s(0) = 1$. Is it possible for these polynomials to be linearly independent?