1. For each of the following statements, write the word "true" or "false."

   (a) Let $f(x)$ be an odd function and $g(x)$ be an even function. Then $f(x)g(x)$ is an odd function.
   True.

   (b) Let $f(x)$ be an odd function and $g(x)$ be an even function. Then $f(x)+g(x)$ is an odd function.
   False. For a counterexample, let $f(x)=x$ and $g(x)=1$.

2. Find the Fourier series for the function $f(x)$ defined by

$$f(x)=\begin{cases}-x^2+\pi^2 & \text{if } -\pi\le x\le 0\\ x^2-\pi^2 & \text{if } 0<x<\pi\end{cases}$$

and $f(x+2\pi)=f(x)$ for any x.

The function $f(x)$ is odd, so each $a_m=0$. We only need to compute the coefficients b_m, and

$$b_m=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(mx)\,dx=\frac{2}{\pi}\int_0^{\pi}(x^2-\pi^2)\sin(mx)\,dx.$$

Using integration by parts twice, we obtain

$$\begin{aligned}\int(x^2-\pi^2)\sin(mx)\,dx&=-\frac{(x^2-\pi^2)\cos(mx)}{m}+\frac{2}{m}\int x\cos(mx)\,dx\\&=-\frac{(x^2-\pi^2)\cos(mx)}{m}+\frac{2x\sin(mx)}{m^2}-\frac{2}{m^2}\int\sin(mx)\,dx\\&=-\frac{(x^2-\pi^2)\cos(mx)}{m}+\frac{2x\sin(mx)}{m^2}+\frac{2\cos(mx)}{m^3}+C.\end{aligned}$$

Then we have that

$$\begin{aligned}b_m&=\frac{2}{\pi}\left(-\frac{(x^2-\pi^2)\cos(mx)}{m}+\frac{2x\sin(mx)}{m^2}+\frac{2\cos(mx)}{m^3}\right)\Bigg|_{x=0}^{\pi}\\&=\frac{2}{\pi}\left(\frac{2\cos(m\pi)}{m^3}-\frac{\pi^2}{m}-\frac{2}{m^3}\right)\\&=\frac{2}{m^3\pi}\left(2(-1)^m-m^2\pi^2-2\right).\end{aligned}$$

Therefore the Fourier series for $f(x)$ is

$$f(x)=\sum_{m=1}^{\infty}\frac{2}{m^3\pi}\left(2(-1)^m-2-m^2\pi^2\right)\sin(mx).$$

This expression can be simplified (or made more complex, depending on how you look at it) by splitting the sum up into odd and even terms:

$$\begin{aligned}f(x)&=\sum_{m=1}^{\infty}b_{2m-1}\sin((2m-1)x)+\sum_{m=1}^{\infty}b_{2m}\sin(2mx)\\&=-\frac{2}{\pi}\left[\sum_{m=1}^{\infty}\frac{4+(2m-1)^3\pi}{(2m-1)^3}\sin((2m-1)x)+\sum_{m=1}^{\infty}\frac{\pi^2}{m}\sin(2mx)\right].\end{aligned}$$