1. Give the general solution to the following system of differential equations.

$$\mathbf{x}' = \begin{pmatrix} -2 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$$

The characteristic polynomial of the matrix is

$$\begin{vmatrix} -2-\lambda & -1 & 0 \\ 0 & -1-\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = -(\lambda+2)\left[(\lambda)(\lambda+1)+1\right]+1 = -(\lambda+1)^3.$$

Then we have that

$$A+I = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

which has a one-dimensional null space. Then our guess can be any vector not in the null space of

$$(A+I)^2 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

such as $\mathbf{v}_1 = (0,0,1)^T$. Then let $\mathbf{v}_2 = (A+I)\mathbf{v}_1 = (0,-1,1)^T$ and $\mathbf{v}_3 = (A+I)\mathbf{v}_2 = (1,-1,0)^T$. Then the general solution takes the form

$$\begin{aligned}
\mathbf{x}(t) &= c_1 \exp(At)\mathbf{v}_1 + c_2 \exp(At)\mathbf{v}_2 + c_3 \exp(At)\mathbf{v}_3 \\
&= e^{-t}\left[c_1\left(I + t(A+I) + \frac{t^2}{2}(A+I)^2\right)\mathbf{v}_1 + c_2\left(I + t(A+I)\right)\mathbf{v}_2 + c_3\mathbf{v}_3\right] \\
&= e^{-t}\left[c_1\left(\mathbf{v}_1 + t\mathbf{v}_2 + \frac{t^2}{2}\mathbf{v}_3\right) + c_2\left(\mathbf{v}_2 + t\mathbf{v}_3\right) + c_3\mathbf{v}_3\right] \\
&= e^{-t}\left[c_1\left(\begin{pmatrix}0\\0\\1\end{pmatrix} + t\begin{pmatrix}0\\-1\\1\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix}\right) + c_2\left(\begin{pmatrix}0\\-1\\1\end{pmatrix} + t\begin{pmatrix}1\\-1\\0\end{pmatrix}\right) + c_3\begin{pmatrix}1\\-1\\0\end{pmatrix}\right].
\end{aligned}$$

2. Give the general solution to the following system of differential equations.

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ -1 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{x}$$

The characteristic polynomial of the matrix is

$$\begin{vmatrix} -\lambda & 1 & 1 \\ -1 & -2-\lambda & -1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = -(\lambda+1)\left[\lambda(\lambda+2)+1\right] = -(\lambda+1)^3.$$

Then we have that

$$A+I = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

which has a null space spanned by $\mathbf{v}_1 = (1, -1, 0)^T$ and $\mathbf{v}_2 = (1, 0, -1)^T$. We can let $\mathbf{v}_3$ be any vector not in the null space, such as $(0, 0, 1)^T$. Note that $(A + I)\mathbf{v}_3 = \mathbf{v}_1$. Then the general solutions takes the form

$$\begin{aligned}
\mathbf{x}(t) &= c_1 \exp(At)\mathbf{v}_1 + c_2 \exp(At)\mathbf{v}_2 + c_3 \exp(At)\mathbf{v}_3 \\
&= e^{-t} \left[c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3 \left(I + t(A + I)\right) \mathbf{v}_3\right] \\
&= e^{-t} \left[c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3 \left(\mathbf{v}_3 + t\mathbf{v}_1\right)\right] \\
&= e^{-t} \left[c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right) \right].
\end{aligned}$$