1. Write the word "true" or "false": Let A be a 3×3 matrix with real entries and complex eigenvalue $\alpha + \beta i$ (that is, α is a real number and β is a nonzero real number). Then A has three distinct eigenvalues.

   True. The roots of the characteristic polynomial of A must be $\alpha + \beta i$, $\alpha - \beta i$, and some real number.

2. (a) Solve the following initial value problem:

$$\mathbf{x}' = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \mathbf{x}$$
$$\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The characteristic polynomial of the matrix is

$$(\lambda - 2)^2 - 16 = \lambda^2 - 4\lambda - 12 = (\lambda + 2)(\lambda - 6),$$

so the eigenvalues are -2 and 6. The eigenspace for -2 is the null space of

$$\begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix}$$

which is spanned by $(1, -1)^T$. The eigenspace for 6 is the null space of

$$\begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$$

which is spanned by $(1, 1)^T$. Therefore the general solution to the system of differential equations is given by

$$y = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t}.$$

From the initial condition, we have that

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and therefore $c_1 = -\frac{1}{2}$ and $c_2 = \frac{1}{2}$. Therefore the solution to the initial value problem is given by

$$y = -\frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t}.$$

(b) What type of system is the above differential equation? Circle one.

   i. source
   ii. sink
   iii. saddle point
   iv. spiral in
   v. spiral out
   vi. periodic

Since the eigenvalues are real and have opposite signs, it is a saddle point.