

1. For each of the following statements, write the word “true” or “false.”

(a) Let P be the matrix corresponding to some orthogonal projection. Then $P^5 = P^3$.

True. Since P is an orthogonal projection, $P^2 = P$. Therefore $P^5 = (P^2)(P^2)(P) = (P)(P)(P) = P^3$.

(b) Let A be a matrix with orthogonal columns. Then $A^T A = I$.

False. The matrix $2I$ has orthogonal columns, but $(2I)^T(2I) = 4I \neq I$.

2. Let $\mathbf{c}_1 = (1, 1, 1)$ and $\mathbf{c}_2 = (2, 0, 1)$. Find the matrix for the orthogonal projection onto the subspace of \mathbb{R}^3 spanned by \mathbf{c}_1 and \mathbf{c}_2 .

First, we need to give an orthogonal basis for the subspace spanned by \mathbf{c}_1 and \mathbf{c}_2 . To do this, use the Gram-Schmidt process. Let $\mathbf{p}_1 = \mathbf{c}_1$. Then let

$$\mathbf{p}_2 = \mathbf{c}_2 - \frac{\langle \mathbf{c}_2, \mathbf{p}_1 \rangle}{\langle \mathbf{p}_1, \mathbf{p}_1 \rangle} \mathbf{p}_1 = (2, 0, 1) - \frac{3}{3}(1, 1, 1) = (1, -1, 0).$$

Then for any $\mathbf{v} \in \mathbb{R}^3$, the orthogonal projection is given by

$$P(\mathbf{v}) = \frac{\langle \mathbf{v}, (1, 1, 1) \rangle}{3} (1, 1, 1) + \frac{\langle \mathbf{v}, (1, -1, 0) \rangle}{2} (1, -1, 0).$$

Therefore,

$$P(1, 0, 0) = \frac{1}{3}(1, 1, 1) + \frac{1}{2}(1, -1, 0) = \left(\frac{5}{6}, -\frac{1}{6}, \frac{1}{3} \right)$$

$$P(0, 1, 0) = \frac{1}{3}(1, 1, 1) - \frac{1}{2}(1, -1, 0) = \left(-\frac{1}{6}, \frac{5}{6}, \frac{1}{3} \right)$$

$$P(0, 0, 1) = \frac{1}{3}(1, 1, 1) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right).$$

Therefore, the matrix corresponding to the orthogonal projection is

$$\frac{1}{6} \begin{pmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$