The sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

is called the sequence of Fibonacci numbers. It is defined by setting $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for any $n \geq 2$.

1. Let

   $$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

   and for each $n \geq 1$, let

   $$\mathbf{a}_n = \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}.$$

   Show that the matrix equations

   $$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

   and $\mathbf{a}_{n+1} = A\mathbf{a}_n$ for each $n \geq 1$ also define the sequence of Fibonacci numbers.

2. Compute the eigenvalues of A. Call them λ_1 and λ_2. Also compute eigenvectors for A. Let $\mathbf{v}_1$ be an eigenvector for λ_1, and let $\mathbf{v}_2$ be an eigenvector for λ_2.

3. Let P be the 2×2 matrix with $\mathbf{v}_1$ as its first column and $\mathbf{v}_2$ as its second. Also, let

   $$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

   Compute PDP^{-1}.

4. Compute A^k for any $k \geq 1$. Your answer should be a 2×2 matrix where the entries depend on k. [Hint: Use the previous problem, and simplify the expression you get before attempting to figure out the entries of the matrix explicitly.]

5. Show that $\mathbf{a}_n = A^{n-1}\mathbf{a}_1$.

6. Compute $\mathbf{a}_n$ for any $n \geq 1$. Your answer should be a vector with two entries, each of which depend on n.

7. Give an explicit formula for a_n in terms of n.

8. Compute

   $$\lim_{n\to\infty} \frac{a_n}{a_{n-1}}.$$

   Do you know what this number is called?