1. (Srping 1985 and Fall 1989 Prelim) Let A be an $n \times n$ matrix and A^T its transpose. Show that $CS(A^T A) = CS(A^T)$ in the following steps.

   (a) Show that $CS(A^T A)$ is a subspace of $CS(A^T)$.

   (b) Show that dim $CS(A^T A) =$ dim $CS(A^T)$. [Hint: Use Theorem 4.18b in section 4.2.]

   (c) Conclude that $CS(A^T A) = CS(A^T)$. (Why?)

2. (Spring 1997 Prelim) Suppose P and Q are $n \times n$ matrices such that $P^2 = P$, $Q^2 = Q$, and $1 - P - Q$ is invertible. Show that P and Q have the same rank. [Hint: Look at the matrices $P(1 - P - Q)$ and $(1 - P - Q)Q$. What effect does multiplication by some invertible matrix have on the rank of a matrix?]

3. (Spring 2001 Prelim) Let A be an $n \times n$ matrix. Prove that

$$rk(A^2) - rk(A^3) \leq rk(A) - rk(A^2).$$

4. (Fall 1998 Prelim) Let A and B be $n \times n$ matrices. Show that dim $NS(AB) \leq$ dim $NS(A) +$ dim $NS(B)$ in the following steps.

   (a) Show that $NS(B) \subseteq NS(AB)$.

   (b) Start with a basis for $NS(B)$ and expand it to a basis for $NS(AB)$. Show that the number of vectors that must be added is at most dim $NS(A)$.

   (c) Show that dim $NS(AB) \leq$ dim $NS(A) +$ dim $NS(B)$.