Consider the system of differential equations

$$\mathbf{x}(t) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix} \mathbf{x}(t). \tag{1}$$

Then the characteristic equation of the above matrix, which we will call A, is

$$\chi_A(\lambda) = -(\lambda - 1)(\lambda^2 + 1) = -(\lambda - 1)(\lambda - i)(\lambda + i).$$

An eigenvector for $\lambda = 1$ is $\mathbf{v}_1 = (1, 1, 1)^T$, so our first solution to (1) is

$$\mathbf{x}_1(t) = e^t \mathbf{v}_1 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}$$

For $\lambda = -i$, we're looking for a vector in the null space of

$$\begin{pmatrix} i & 0 & 1 \\ 1 & i & 0 \\ -1 & 1 & 1+i \end{pmatrix}$$

such as $(-1, -i, i)^T$. Then we have a complex-valued solution to (1) given by

$$e^{-it} \begin{pmatrix} -1 \\ -i \\ i \end{pmatrix} = (\cos t - i \sin t) \begin{pmatrix} -1 \\ -i \\ i \end{pmatrix} = \begin{pmatrix} -\cos t \\ -\sin t \\ \sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ -\cos t \\ \cos t \end{pmatrix}.$$

The real and imaginary parts of this complex-valued solution give us the following real-valued solutions to (1):

$$\mathbf{x}_2(t) = \begin{pmatrix} -\cos t \\ -\sin t \\ \sin t \end{pmatrix}$$

$$\mathbf{x}_3(t) = \begin{pmatrix} \sin t \\ -\cos t \\ \cos t \end{pmatrix}.$$

Since we now have three linearly independent solutions to (1), we can use them as the columns of a matrix,

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^t & -\cos t & \sin t \\ e^t & -\sin t & -\cos t \\ e^t & \sin t & \cos t \end{pmatrix}.$$

Then the fundamental matrix is given by $\mathbf{\Phi}(t) = \mathbf{\Psi}(t) \left(\mathbf{\Psi}(0)\right)^{-1}$, where we are assuming that the initial condition will be given for $t = 0$. Since

$$\mathbf{\Psi}(0) = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

we have that

$$\left(\mathbf{\Psi}(0)\right)^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and therefore

$$\mathbf{\Phi}(t) = \frac{1}{2}\begin{pmatrix} 2\cos t & e^t - \sin t - \cos t & e^t + \sin t - \cos t \\ 2\sin t & e^t - \sin t + \cos t & e^t - \sin t - \cos t \\ -2\sin t & e^t + \sin t - \cos t & e^t + \sin t + \cos t \end{pmatrix}.$$