1. Show that the curve defined by the polar equation $r = e^\theta$ is such that the angle between the curve and the radial line is the same at each point.

   It is enough to show that the cosine of that angle is the same at each point. The angle we are asked about is the same as the angle between the radial vector (usually denoted $\mathbf{r}$, the vector from the origin to the point on the curve) and the tangent vector.

   Polar coordinates may be a concise way to describe this spiral, but they aren't particularly helpful for solving this problem. So, the first thing we want to do is dump polar coordinates. Recall the equations used to convert to cartesian coordinates,

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

   For this curve, we obtain

$$x = e^\theta\cos\theta$$
$$y = e^\theta\sin\theta.$$

   Then we can write the curve in vector form,

$$\mathbf{r}(\theta) = \langle e^\theta\cos\theta, e^\theta\sin\theta\rangle.$$

   So that's the radial vector. Next we need to compute the tangent vector:

$$\mathbf{r}'(\theta) = \langle e^\theta(\cos\theta - \sin\theta), e^\theta(\sin\theta + \cos\theta)\rangle.$$

   We have been asked about the angle between the radial and tangent vectors or, equivalently, its cosine. Recall the formula for computing the cosine of the angle between two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ using the dot product,

$$\cos\phi = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|},$$

   where $\phi$ is the angle between the two vectors. Applying this formula to $\mathbf{r}$ and $\mathbf{r}'$, we obtain

$$\begin{aligned}\cos\phi &= \frac{\mathbf{r}(\theta)\cdot\mathbf{r}'(\theta)}{|\mathbf{r}(\theta)||\mathbf{r}'(\theta)|}\\ &= \frac{e^{2\theta}(\cos^2\theta - \sin\theta\cos\theta + \sin^2\theta + \sin\theta\cos\theta)}{(e^\theta)(e^\theta\sqrt{2})}\\ &= \frac{1}{\sqrt{2}},\end{aligned}$$

   which is constant with respect to $\theta$.

2. Suppose a curve $\mathbf{r}(t)$ is such that its tangent and radial vectors are always perpendicular. What can be said about such a curve?

   Let $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ for all $t \in \mathbb{R}$. Then we are given that $f(t)f'(t) + g(t)g'(t) = \mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$ for all $t$. Then

$$\begin{aligned}\frac{d|\mathbf{r}(t)|^2}{dt} &= \frac{d}{dt}\left[(f(t))^2 + (g(t))^2\right]\\ &= 2f(t)f'(t) + 2g(t)g'(t)\\ &= 2\left[f(t)f'(t) + g(t)g'(t)\right]\\ &= 0\end{aligned}$$

for all $t$. As long as we assume that $\mathbf{r}(t)$ is differentiable everywhere, this means that the length squared of $\mathbf{r}(t)$ is constant, which means that the length of $\mathbf{r}(t)$ is constant. Therefore, the curve lies entirely along some circle centered around the origin.

3. Suppose we are given spherical coordinates for two points on a sphere of radius $R$ centered around the origin: $(R, \theta_1, \phi_1)$ and $(R, \theta_2, \phi_2)$. We wish to find the *great circle distance* between them.

   First, we need to know the angle made by the radial vectors of these two points. We can find the cosine of this angle by taking the dot product of the two unit vectors in the same direction: $\langle \sin\phi_1 \cos\theta_1, \sin\phi_1 \sin\theta_1, \cos\phi_1 \rangle$ and $\langle \sin\phi_2 \cos\theta_2, \sin\phi_2 \sin\theta_2, \cos\phi_2 \rangle$. Their dot product is

$$\sin\phi_1 \sin\phi_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) + \cos\phi_1 \cos\phi_2 = \sin\phi_1 \sin\phi_2 \cos(\theta_1 - \theta_2) + \cos\phi_1 \cos\phi_2.$$

   Once we have the angle between the two vectors in radians, the great circle distance is given by the radius of the sphere times the measure of that angle. So, the great circle distance between $(R, \theta_1, \phi_1)$ and $(R, \theta_2, \phi_2)$ is

$$R \cos^{-1} \left( \sin\phi_1 \sin\phi_2 \cos(\theta_1 - \theta_2) + \cos\phi_1 \cos\phi_2 \right).$$