Quiz 11 Solutions

1. Let f and g be scalar functions of three variables with continuous second-order partial derivatives. Compute $\operatorname{div}(\nabla f \times \nabla g)$. To get full credit on this problem, you must simplify completely. (This is exercise 28 in section 16.5. Exercises 23-29 in that section asked you to prove various identities involving divergence, curl, and the Laplacian. If you can remember what any of those exercies *other* than number 28 say, you may use them without proving them.)

   If you don't remember any of those other exercises, you can compute the divergence as follows. The conditions of Clairaut's Theorem are satisfied for f and g, and we use that theorem at the end of the computation to put all the subscripts in alphabetical order.

$$\begin{aligned}
\operatorname{div}(\nabla f \times \nabla g) &= \operatorname{div}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} \\
&= \frac{\partial}{\partial x}(f_y g_z - f_z g_y) + \frac{\partial}{\partial y}(f_z g_x - f_x g_z) + \frac{\partial}{\partial z}(f_x g_y - f_y g_x) \\
&= f_y g_{zx} + f_{yx} g_z - f_z g_{yx} - f_{zx} g_y \\
&\quad + f_z g_{xy} + f_{zy} g_x - f_x g_{zy} - f_{xy} g_z \\
&\quad + f_x g_{yz} + f_{xz} g_y - f_y g_{xz} - f_{yz} g_x \\
&= f_y g_{xz} + f_{xy} g_z - f_z g_{xy} - f_{xz} g_y \\
&\quad + f_z g_{xy} + f_{yz} g_x - f_x g_{yz} - f_{xy} g_z \\
&\quad + f_x g_{yz} + f_{xz} g_y - f_y g_{xz} - f_{yz} g_x \\
&= 0.
\end{aligned}$$

   However, an easier way to do this is to remember exercise 27, which says

$$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}.$$

   Letting $\mathbf{F} = \nabla f$ and $\mathbf{G} = \nabla g$, this gives us that $\operatorname{div}(\nabla f \times \nabla g) = 0$, since the curl of any gradient field is $\mathbf{0}$.

2. Find the flux of the vector field $\mathbf{F}(x, y, z) = \frac{x}{(x^2+y^2+z^2)^{3/2}} \cdot \mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}} \cdot \mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}} \cdot \mathbf{k}$ through the surface $x^2 + y^2 + z^2 = 1$, oriented outward.

   Note that $\mathbf{F}$ always points in the same direction as the normal to this surface. So, at any point on the surface,

$$\begin{aligned}
\mathbf{F}(x, y, z) \cdot \mathbf{n} &= |\mathbf{F}(x, y, z)| \cdot |\mathbf{n}| \\
&= \sqrt{\frac{x^2}{(x^2+y^2+z^2)^3} + \frac{y^2}{(x^2+y^2+z^2)^3} + \frac{z^2}{(x^2+y^2+z^2)^3}} \\
&= \sqrt{\frac{1}{(x^2+y^2+z^2)^2}} \\
&= 1.
\end{aligned}$$

   Therefore, if we call the given surface D, the flux integral is given by

$$\int_D \mathbf{F}\, d\mathbf{S} = \int_D \mathbf{F} \cdot \mathbf{n}\, dS = \int_D dS,$$

   which is the surface area of D. This is 4π.