Quiz 10 Solutions

1. Let $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$ and $f(x,y) = 3x^2 + y^2 + \sqrt{x^2+y^2} + \frac{y^2}{\sqrt{x^2+y^2}}$. Compute $\int\int_D f(x,y)\,dA$.

   Let $P(x,y) = -y\sqrt{x^2+y^2}$ and $Q(x,y) = x(x^2+y^2)$. Then $\frac{\partial P}{\partial y} = -\left(\sqrt{x^2+y^2} + \frac{y^2}{\sqrt{x^2+y^2}}\right)$ and $\frac{\partial Q}{\partial x} = 3x^2+y^2$. So, $f(x,y) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. By Green's Theorem, if C is the counterclockwise unit circle, $\int\int_D f(x,y)\,dA = \int_C P\,dx + Q\,dy$. C can be parametrized by $x(t) = \cos t$, $y(t) = \sin t$ for $0 \le t \le 2\pi$. So,

$$\begin{aligned}\int_C P\,dx + Q\,dy &= \int_C x\,dy - y\,dx \\ &= \int_0^{2\pi} \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\,dt \\ &= \int_0^{2\pi} \left(\cos^2 t + \sin^2 t\right)\,dt \\ &= 2\pi.\end{aligned}$$

2. Find the curl and divergence of the vector field $\mathbf{F}(x,y,z) = \langle xe^{-y}, xz - e^y + e^{-y}, ze^y\rangle$. Please clearly label both answers.

$$\begin{aligned}\text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^{-y} & xz - e^y + e^{-y} & ze^y \end{vmatrix} \\ &= (ze^y - x)\mathbf{i} - (0-0)\mathbf{j} + (z + xe^{-y})\mathbf{k} \\ &= (ze^y - x)\mathbf{i} + (z + xe^{-y})\mathbf{k}.\end{aligned}$$

$$\begin{aligned}\text{div } \mathbf{F} &= \frac{\partial}{\partial x}(xe^{-y}) + \frac{\partial}{\partial y}(xz - e^y + e^{-y}) + \frac{\partial}{\partial z}(ze^y) \\ &= e^{-y} - e^y - e^{-y} + e^y \\ &= 0.\end{aligned}$$