Quiz 9 Solutions

1. Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy$. [Hint: change to cylindrical coordinates.]

   The region given consists of the part of the circle in the first quadrant in the xy-plane, and goes from r^2 to r in the z direction. So, our integral is

$$\begin{aligned}
\int_0^1 \int_0^{\frac{\pi}{2}} \int_{r^2}^{r} r^3 \sin\theta \cos\theta z \, dz \, d\theta \, dr &= \frac{1}{4} \int_0^1 \int_0^{\frac{\pi}{2}} r^3 \sin(2\theta)(r^2 - r^4) \, d\theta \, dr \\
&= \frac{1}{4} \int_0^1 r^5 - r^7 \, dr \int_0^{\frac{\pi}{2}} \sin(2\theta) \, d\theta \\
&= \frac{1}{4} \left(\frac{1}{6} - \frac{1}{8} \right) \\
&= \frac{1}{96}.
\end{aligned}$$

2. Find the area of the region bounded by $xy = 1$, $xy = 8$, $y = x^2$, and $y = 8$.

   Let $x = u$ and $y = \frac{v}{u}$. Then our bounding curves become $v = 1$, $v = 8$, $v = u^3$, and $v = 8u$. Also, $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{u}$. So, the area of the region is

$$\begin{aligned}
\int_1^8 \int_{\frac{v}{8}}^{\sqrt[3]{v}} \frac{1}{u} \, du \, dv &= \int_1^8 \ln \sqrt[3]{v} - \ln(v/8) \, dv \\
&= \int_1^8 \ln 8 - \frac{2}{3} \ln v \, dv \\
&= \left(v \ln 8 - \frac{2}{3}(v \ln v - v) \right) \Big|_{v=1}^{8} \\
&= 8 \ln 8 - \frac{16}{3} \ln 8 + \frac{16}{3} - \ln 8 - \frac{2}{3} \\
&= 5 \ln 2 + \frac{14}{3}.
\end{aligned}$$