Quiz 8 Solutions

1. Find the surface area of the part of the ellipsoid $x^2 + y^2 + 5z^2 = 9$ above the plane $z = 1$.

   This was a botched problem. Full credit for everyone who took the quiz.

   The intersection of the ellipsoid and the plane is given by $x^2 + y^2 = 4$. So, we are trying to find the surface area of the function $z = \sqrt{\frac{9-x^2-y^2}{5}}$ over the region $x^2+y^2 \leq 4$. We have that $\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{5(9-x^2-y^2)}}$ and $\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{5(9-x^2-y^2)}}$. Therefore the surface area is

$$\int\int_{x^2+y^2\leq 4} \sqrt{1 + \frac{x^2}{5(9-x^2-y^2)} + \frac{y^2}{5(9-x^2-y^2)}}\, dA = \int\int_{x^2+y^2\leq 4} \sqrt{\frac{45-4(x^2+y^2)}{5(9-(x^2+y^2))}}\, dA.$$

   In polar coordinates, this is

$$\begin{aligned}\int_0^{2\pi}\int_0^2 r\sqrt{\frac{45-4r^2}{45-5r^2}}\, dr\, d\theta &= 2\pi \int_0^2 r\sqrt{\frac{45-4r^2}{45-5r^2}}\, dr \\ &= \frac{\pi}{10}\left[90 - 10\sqrt{29} + 9\sqrt{5}\sinh^{-1} 2 + 9\sqrt{5}\ln 3 - 9\sqrt{5}\ln(2\sqrt{5}+\sqrt{29})\right] \\ &\approx 12.9624.\end{aligned}$$

2. Evaluate the integral $\int_0^1 \int_x^2 \int_0^y x^2yz\, dz\, dy\, dx$.

$$\begin{aligned}\int_0^1\int_x^2\int_0^y x^2yz\, dz\, dy\, dx &= \frac{1}{2}\int_0^1\int_x^2 x^2y^3\, dy\, dx \\ &= \frac{1}{8}\int_0^1 x^2(16-x^4)\, dx \\ &= \frac{1}{8}\left(\frac{16}{3} - \frac{1}{7}\right) \\ &= \frac{109}{168}.\end{aligned}$$