Quiz 7 Solutions

1. Does Fubini's Theorem apply to the following integrals? Why or why not? You do not need to evaluate the integrals, but you should state the conditions of Fubini's Theorem and say whatever you can about why those conditions either are or are not satisfied for each integral.

   (a) $\int_0^1 \int_0^1 \frac{1}{x+y}\, dx\, dy$

   The function $\frac{1}{x+y}$ grows asymptotically as (x, y) goes to $(0, 0)$. Therefore this function is not bounded on $[0, 1] \times [0, 1]$, so Fubini's Theorem does not apply.

   (b) $\int_0^1 \int_0^1 \frac{\sin(x+y)}{x+y}\, dx\, dy$

   The function is continuous if we define it to be 1 at $(0, 0)$. Therefore it is bounded, discontinuous on only a finite number of smooth curves, and the iterated integrals exist; so Fubini's Theorem applies.

   In general, the iterated integrals of a function f on a region R will exist whenever $\int \int_R |f|\, dA < \infty$. This is always true if f is bounded and R has finite area.

2. Find the volume of the solid under the surface $z = xy^2$ and above the region bounded by $y = x^2$ and $x = y^2$.

$$\begin{aligned}
\int_0^1 \int_{x^2}^{\sqrt{x}} xy^2\, dy\, dx &= \int_0^1 x \left(\frac{y^3}{3}\right)\Big|_{y=x^2}^{\sqrt{x}}\, dx \\
&= \frac{1}{3}\int_0^1 x\left(x^{3/2} - x^6\right)\, dx \\
&= \frac{1}{3}\left(\frac{2}{7}x^{7/2} - \frac{x^8}{8}\right)\Big|_{x=0}^{1} \\
&= \frac{1}{3}\left(\frac{2}{7} - \frac{1}{8}\right) \\
&= \frac{1}{3}\cdot\frac{9}{56} \\
&= \frac{3}{56}.
\end{aligned}$$