Quiz 6 Solutions

1. Find the points on the surface $x^2yz = 1$ that are closest to the origin.

   Solution 1. For a point (x, y, z) on this surface, we can write its distance squared from the origin, D, as a function of y and z:

   $$D(y,z) = y^2 + z^2 + \frac{1}{yz}.$$

   We use distance squared instead of distance because this simplifies the formula slightly, and minimizing distance squared the is the same as minimizing distance. Then we need to compute the gradient of D:

   $$\begin{aligned}\nabla D(y,z) &= \left\langle 2y - \frac{1}{y^2z}, 2z - \frac{1}{yz^2} \right\rangle \\ &= \left\langle \frac{2y^3z-1}{y^2z}, \frac{2yz^3-1}{yz^2} \right\rangle.\end{aligned}$$

   So, $\nabla D(y,z) = \mathbf{0}$ if and only if $2y^3z = 1 = 2yz^3$, which implies $y^2 = z^2$, which is in turn equivalent to $y = z$ or $y = -z$. However, since $x^2yz > 0$, $yz > 0$, so $\nabla D(y,z) = \mathbf{0}$ only if $y = z$. Then $\nabla D(y,z) = \mathbf{0}$ if and only if $2y^4 = 1$, which is equivalent to $y = \frac{1}{\sqrt[4]{2}}$ or $y = -\frac{1}{\sqrt[4]{2}}$. So, there are four points on the surface with minimum distance from the origin: $\left(\sqrt[4]{2}, \frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}\right)$, $\left(-\sqrt[4]{2}, \frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}\right)$, $\left(\sqrt[4]{2}, -\frac{1}{\sqrt[4]{2}}, -\frac{1}{\sqrt[4]{2}}\right)$, and $\left(-\sqrt[4]{2}, -\frac{1}{\sqrt[4]{2}}, -\frac{1}{\sqrt[4]{2}}\right)$.

   Solution 2. Write D as a function of x as well: $D(x,y,z) = x^2 + y^2 + z^2$. Then we wish to maximize D under the constraint $x^2yz = 1$. So, $\nabla D(x,y,z) = \langle 2x, 2y, 2z \rangle$, and if we set $g(x,y,z) = x^2yz$, then $\nabla g(x,y,z) = \langle 2xyz, x^2z, x^2y \rangle$. Then, if $\lambda \nabla D(x,y,z) = \nabla g(x,y,z)$, then $\lambda = yz$ by looking at the first coordinate, so $x^2 = 2y^2$ and $x^2 = 2z^2$ from looking at the other coordinates. Then we know that $y^2 = z^2$, and as above, since $x^2yz > 0$, $yz > 0$, so $y = z$. Then, since $x^2 = 2y^2$, $y = z$ and $x^2yz = 1$, $2y^4 = 1$. The rest of the solution follows as above.

2. Find the maximum and minimum values of the function $f(x,y) = 5x + 12y$ subject to the constraint $x^2 + y^2 = 4$.

   Let $g(x,y) = x^2 + y^2$. Then $\nabla f(x,y) = \langle 5, 12 \rangle$ and $\nabla g(x,y) = \langle 2x, 2y \rangle$. These vectors will be scalar multiples of one another if and only if $y = \frac{12}{5}x$. The only such points lying on the curve $g(x,y) = 4$ are $\left(\frac{10}{13}, \frac{24}{13}\right)$ and $\left(-\frac{10}{13}, -\frac{24}{13}\right)$. So, $f\left(\frac{10}{13}, \frac{24}{13}\right) = 26$ is the maximum value of f under the constraint, and $f\left(-\frac{10}{13}, -\frac{24}{13}\right) = -26$ is the minimum value.