Quiz 3 Solutions

1. (a) Prove the *Cauchy-Schwarz Inequality*:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then both sides of the inequality are 0. Otherwise, let $\theta$ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then,

$$\begin{aligned} |\mathbf{a} \cdot \mathbf{b}| &= ||\mathbf{a}||\mathbf{b}| \cos\theta| \\ &= |\mathbf{a}||\mathbf{b}| \cdot |\cos\theta| \\ &\leq |\mathbf{a}||\mathbf{b}|. \end{aligned}$$

(b) Prove the *Triangle Inequality*:

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

It will suffice to show that $|\mathbf{a} + \mathbf{b}|^2 \leq (|\mathbf{a}| + |\mathbf{b}|)^2$. By the properties of the dot product and part (a),

$$\begin{aligned} |\mathbf{a} + \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) \\ &= |\mathbf{a}|^2 + 2(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2 \\ &\leq |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 \\ &= (|\mathbf{a}| + |\mathbf{b}|)^2. \end{aligned}$$

2. Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

By the properties of the cross product,

$$\begin{aligned} (\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) &= \mathbf{a} \times \mathbf{a} - \mathbf{b} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{b} \\ &= 0 + \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{b} - 0 \\ &= 2(\mathbf{a} \times \mathbf{b}). \end{aligned}$$