Quiz 2 Solutions

1. Find the area bounded by the curve $x = t - 4/t$, $y = t + 4/t$ and the line $y = 5$.

   The parametric curve is the hyperbola $y^2 - x^2 = 16$, the upper half of which intersects the line $y = 5$ at $(-3, 5)$ and $(3, 5)$, corresponding to $t = 1$ and $t = 4$, respectively. Since the line is parametrized by $x = 2t - 5$ and $y = 5$, the area bounded by these two curves is given by

$$\begin{aligned}
\int_1^4 10 - \left(1 + \frac{4}{t^2}\right)\left(t + \frac{4}{t}\right)\, dt &= 30 - \int_1^4 \frac{t^2+4}{t^2}\frac{t^2+4}{t}\, dt \\
&= 30 - \int_1^4 \frac{t^4 + 8t^2 + 16}{t^3}\, dt \\
&= 30 - \int_1^4 t + \frac{8}{t} + \frac{16}{t^3}\, dt \\
&= 30 - \left(\frac{t^2}{2} + 8\ln|t| - \frac{8}{t^2}\right)\Big|_{t=1}^4 \\
&= 30 - 8 - 8\ln 4 + \frac{1}{2} + \frac{1}{2} - 8 \\
&= 15 - 8\ln 4.
\end{aligned}$$

2. Find all points of intersection of the two curves $r = \sin\theta$ and $r = \cos\theta$.

   Since r is periodic with period 2π for both curves, we may assume $0 \le \theta < 2\pi$. Suppose (x, y) is some point of intersection in cartesian coordinates. Then, for some θ_1 and θ_2,

$$x = \sin\theta_1 \cos\theta_1 = \cos^2\theta_2 \tag{1}$$

   and

$$y = \sin^2\theta_1 = \sin\theta_2\cos\theta_2. \tag{2}$$

   There are three ways for points with polar coordinates to coincide. One is for their radii and angles to be equal. The second is for their angles to be offset by π but with oppositve radii. The last is for both to have radii 0 and any angle whatsoever.

   Suppose $\theta_1 = \theta_2$. By (1), either $\cos\theta_1 = 0$ or $\sin\theta_1 = \cos\theta_1$. However, if $\cos\theta_1 = 0$, then by (2) $\sin\theta_1 = 0$. So, $\sin\theta_1 = \cos\theta_1$. This happens only when they are both equal to either $\frac{\sqrt{2}}{2}$ or $-\frac{\sqrt{2}}{2}$. Both cases correspond to the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ in cartesian coordinates.

   Suppose $\theta_2 = \theta_1 + \pi$. Then (1) becomes $\sin\theta_1\cos\theta_1 = \cos^2\theta_1$, and (2) becomes $\sin^2\theta_1 = \sin\theta_1\cos\theta_1$. As before, these equations give a single point of intersection: $\left(\frac{1}{2}, \frac{1}{2}\right)$.

   Both curves have radius 0 at some point, so another point of intersection is $(0, 0)$.

   So, the curves intersect in exactly two places: $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$.